The large-maturity smile for the Stein-Stein model

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Abstract: We compute the large-maturity smile for the correlated Stein-Stein stochastic volatility model $dS_t = S_t Y_t dW_t^1$, $dY_t = \kappa(\theta - Y_t)dt + \sigma dW_t^2$, $dW_t^1 dW_t^2 = \rho dt$, using the known closed-form solution for the characteristic function of the log stock price given in Schöbel&Zhu[SZ99]. The Stein-Stein model is not covered by the results in [FK13] and [JKRM13] because the volatility fails to satisfy the sublinear growth condition in [FK13] and is not an affine model.†

1. Introduction

The last few years have witnessed a number articles on large-time asymptotics for stochastic volatility models with/without a jump component. Using the Gärtner-Ellis theorem, [FJ11] compute the implied volatility smile for the popular Heston stochastic volatility model when $\kappa > 0$, $\kappa > \rho\sigma$, in the large-time, large log-moneyness regime and [FJM10] compute the correction term using saddlepoint methods; the large-time smile is identical to the large-time smile for the Barndorff-Nielsen Normal Inverse Gaussian model, and [GJ11] show that the asymptotic smile can be computed in closed-form via the Gatheral SVI parameterization. [JM12] derive similar results for a displaced Heston model, and relax the aforementioned conditions on κ, ρ, σ. [JKRM13] have extended the results in [FJ11] to a general class of affine stochastic volatility models (with jumps), which includes the Heston model with state-independent jumps, the Bates model with state-dependent jumps and the Barndorff-Nielsen-Shephard model.

[FP12] compute large-time asymptotics for the SABR model with $\beta = 1, \rho \leq 0$ and $\beta < 1, \rho = 0$; in particular for $\beta = 1, \rho \leq 0$, they compute a closed-form expression for the asymptotic log stock price density and establish large-time asymptotics for the CEV model and the uncorrelated CEV-Heston model in the large-time, fixed-strike regime and a new large-time, large log-moneyness regime. [Forde11b] derives similar results for the modified SABR model in terms of the large-time asymptotic density of the Brownian exponential functional.

The long-term asymptotic behavior of the smile for exponential Lévy models and more general martingale models have been studied in [RT10], where it is proved that for fixed log-moneyness k and large maturity, the implied volatility converges to a constant value that does not depend on k. This phenomenon is typically referred as the "smile-flattening" effect, which arises from the large deviation principle for i.i.d. random variables (see e.g. Cramér's theorem in [DZ98]). For a general exponential Lévy model with mild conditions on the cumulant generating function, [GL11] derive an expansion of the form $\hat{\sigma}_t(x)^2 = \sigma_\infty^2 + a_1(x)/t + a_2(x)/t^2 + o((\log t)^2/t^3)$ as $t \to \infty$ for the implied volatility $\hat{\sigma}_t(x)$ at log-moneyness x and maturity t, where $a_1(x)$ and $a_2(x)$ are respectively affine and quadratic in x.

In [Forde11], the author derives a large deviation principle for the log stock price under an uncorrelated stochastic volatility model driven by an Ornstein-Uhlenbeck process with a bounded volatility function. For this we use the fact that the occupation measure for the Ornstein-Uhlenbeck process satisfies an LDP with a good, convex lower semicontinuous rate function under the topology of weak convergence (and under the Prohorov metric), see section 7 in Donsker&Varadhan[DV76] (see also page 178 in Stroock[Str84] and [Pin85]), combined with the standard contraction principle and exponential tightness. In [FK13], we relax the assumptions of bounded volatility and zero correlation made in [Forde11]. The rate function for X_t/t now has the variational representation $I(x) = \inf_{\mu \in \mathcal{P}(\mathbb{R})} \frac{(x - M(\mu))^2}{2\nu(\mu)} + I_\alpha(\mu)$, for some linear functionals M, ν which depend on the correlation ρ. Using the LDP, we translate these results into large-time asymptotics for call options and implied volatility, and we extend the analysis to incorporate stochastic interest rates, by deriving a similar LDP for a three-factor model driven a CIR short rate process.

In this article, we look at the large-time behavior of the closed form expression for the characteristic function of the log stock price under the Stein-Stein model introduced in [SS91], which is derived in [SZ99]. Using the Gärtner-Ellis

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Theorem from large deviations theory, we compute a large-time large deviation principle for the log stock price. From this we can then characterize the large-time behavior of call option prices and implied volatility in the large-time, large log-moneyness regime. The Stein-Stein model reduces to a special case of the Heston model when the mean reversion level $\theta = 0$. We refer the reader to Deuschel et al.[DFJV14] for a discussion on tail asymptotics for the Stein-Stein model using Laplace's method on Wiener space for a small-noise diffusion process and some simple scaling properties, and the earlier work on tail asymptotics for the zero correlation case in Gulisashvili&Stein[GS10].

2. Large deviation theory and the Gärtner-Ellis theorem

In this section, we recall some fundamental notions in large deviations theory (we refer the reader to Section 2.3 in [DZ98] and Section 2.2 in [JM12] for more details). A family of random variables (Z_t) is said to satisfy the large deviation principle (LDP) as $t \to \infty$ with good rate function I if for all $B \in \mathcal{B}(\mathbb{R})$ we have the following bounds

$$-\inf_{x\in B^o} I(x) \quad \leq \quad \liminf_{t\to\infty} \frac{1}{t}\log \mathbb{P}(Z_t \in B) \quad \leq \quad \limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}(Z_t \in B) \quad \leq \quad -\inf_{x\in \bar{B}} I(x)\,,$$

where B^o ($\bar{B}$) denotes the interior (resp. closure) of B.

We now assume that the cumulant generating function $V_t(p) = \log \mathbb{E}(e^{pZ_t})$ is finite on some neighbourhood of zero and that the following limit exists as an extended real number

$$V(p) \quad = \quad \lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}(e^{p\,tZ_t}) \qquad \forall p \in \mathbb{R}\ . \tag{2.1}$$

Let $\mathcal{D}_V = \{p \in \mathbb{R} : |V(p)| < \infty\}$ and assume that $\{0\} \in \mathcal{D}_V^o$. From Hölder's inequality we can show that V_t is convex for all $t > 0$ and the limit V is also convex (see Lemma 2.3.9 in [DZ98]). Moreover $V(0) = 0$, thus (by convexity) we see that $V(p) > -\infty$ for all $p \in \mathbb{R}$. $V : \mathbb{R} \to (-\infty, \infty]$ is called *essentially smooth* if V is differentiable in $\mathcal{D}_V^o$ and satisfies $\lim_{n\to\infty} |V'(p_n)| = \infty$ for every sequence (p_n) in $\mathcal{D}_V^o$ which converges to a boundary point of $\mathcal{D}_V^o$. A cgf V which satisfies this second property is called *steep*. The Fenchel-Legendre transform V^* of V is defined by the variational formula

$$V^*(x) \quad = \quad \sup_{p\in\mathbb{R}}[px \,-\, V(p)]$$

for all $x \in \mathbb{R}$, with an effective domain $\mathcal{D}_{V^*} = \{x \in \mathbb{R} : V^*(x) < \infty\}$. In general V^* can be discontinuous and $\mathcal{D}_{V^*}$ can be a strict subset of $\mathbb{R}$ (see section 2.3 in [DZ98] for some simple examples).

We now state a simplified version of Gärtner-Ellis theorem (c.f. Theorem 2.3.6 in [DZ98]) which will needed in the next section.

Theorem 2.1. *Let $(Z_t)_{t>0}$ be a family of random variables for which V as defined in (2.1) satisfies $\{0\} \in \mathcal{D}_V^o$. If V is essentially smooth and lower semicontinuous, then the LDP holds with good rate function V^*.*

3. The Stein-Stein model

From here on, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ throughout, supporting two independent Brownian motions and satisfying the usual conditions. We now recall the Stein-Stein stochastic volatility model for a log stock or forward price process $X_t = \log S_t$:

$$\begin{cases} dX_t \;=\; -\frac{1}{2}Y_t^2 dt + Y_t dW_t^1\,, \\ dY_t \;=\; \kappa(\theta - Y_t)dt + \sigma dW_t^2 \end{cases} \tag{3.1}$$

where $\kappa, \sigma > 0, X_0 = x_0, Y_0 = y_0$, and W^1, W^2 are Brownian motions such that $dW_t^1 dW_t^2 = \rho dt$, $|\rho| < 1$. The law of $X_t - x_0$ does not depend on x_0, so without loss of generality we set $X_0 = 0$.

We first verify the martingale property for S_t.

Proposition 3.1. (S_t) *is a martingale.*

Proof. Let $0 < t_1 < t_2 < \infty$. We know that $\sup_{t\geq 0}\mathbb{E}(e^{cY_t^2}) < \infty$ if $c < \kappa/\sigma^2$, using that $Y_t \sim N(e^{-\kappa t}y_0 + \theta(1-e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1-e^{-2\kappa t}))$. Now consider a uniform random variable U on $[t_1,t_2]$, independent of S, and let $\mathcal{F}_t^Y = \sigma(Y_s; 0 \leq s \leq t)$ denote the filtration generated by the Y process. Then we have

$$\begin{aligned}
\mathbb{E}(e^{\frac{1}{2}\int_{t_1}^{t_2} Y_s^2 dt}) &= \mathbb{E}(e^{\frac{1}{2}(t_2-t_1)\mathbb{E}(Y_U^2 \,|\, \mathcal{F}_t^Y)}) \\
&\leq \mathbb{E}(\mathbb{E}(e^{\frac{1}{2}(t_2-t_1)Y_U^2}|\mathcal{F}_t^Y)) \\
&\quad \text{(using the conditional Jensen's inequality)} \\
&= \mathbb{E}(\frac{1}{t_2-t_1}\int_{t_1}^{t_2} e^{\frac{1}{2}(t_2-t_1)Y_s^2} ds) \;<\; \infty \\
&= \frac{1}{t_2-t_1}\int_{t_1}^{t_2} e^{\frac{1}{2}(t_2-t_1)\mathbb{E}(Y_s^2)} ds) \;<\; \infty \\
&\quad \text{(by Fubini's theorem)}
\end{aligned}$$

for $\frac{1}{2}(t_2-t_1) \leq \kappa/\sigma^2$. By Corollary 5.14, p.199 in [KS91], we conclude that $S_t = e^{-\frac{1}{2}\int_0^t Y_s^2 ds + \int_0^t Y_s dW_s^1}$ is a martingale. □

3.1. The large-time large deviation principle for the re-scaled log return

The following proposition establishes a large-time large deviation principle for the re-scaled log return for the Stein-Stein model:

Proposition 3.2. X_t/t *satisfies a large-time LDP as* $t \to \infty$ *with a good convex continuous rate function given by the Fenchel-Legendre transform*

$$I(x) \;=\; \sup_p [px - V(p)]$$

where

$$V(p) = V(p;\kappa,\theta,\sigma,\rho) \;=\; \begin{cases} \frac{1}{2}\left[\kappa - p\rho\sigma + \frac{(p-1)p\theta^2\kappa^2}{\Gamma(p)^2} - \Gamma(p)\right] & (p \in (p_-,p_+)) \\ +\infty & (p \notin (p_-,p_+)) \end{cases}$$

$\bar{\rho} = \sqrt{1-\rho^2}$, $\Gamma(p) = \sqrt{\kappa^2 - 2p\kappa\rho\sigma + p(1-p\bar{\rho}^2)\sigma^2}$ *and* $p_\pm = \frac{\sigma^2 - 2\kappa\rho\sigma \pm \sigma\sqrt{4\kappa^2 - 4\kappa\rho\sigma + \sigma^2}}{2\sigma^2\bar{\rho}^2}$ *are the roots of* $\Gamma(p)^2$*.* I *is continuous and attains its minimum value uniquely at* $x^* = V'(0) = -\frac{1}{2}(\theta^2 + \sigma^2/2\kappa)$*.*

Proof. From Eq 13 in [SZ99], we have the following closed-form expression for the characteristic function of the log return

$$\phi_t(u) \;=\; \mathbb{E}(e^{iuX_t}) \;=\; e^{-\frac{1}{2}iu\rho(\sigma^{-1}y^2+\sigma t)+\frac{1}{2}D(t,\hat{s}_1,\hat{s}_2,\hat{s}_3)y^2 + B(t,\hat{s}_1,\hat{s}_2,\hat{s}_3)y + C(t,\hat{s}_1,\hat{s}_2,\hat{s}_3)} \tag{3.2}$$

for $u \in \mathbb{R}$, where

$$\begin{aligned}
D(t,T) &= \frac{1}{\sigma^2}[\kappa - \gamma_1 \frac{\sinh(\gamma_1 t) + \gamma_2\cosh(\gamma_1 t)}{\cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t)}] \\
B(t,T) &= \frac{1}{\sigma^2\gamma_1}\frac{\kappa\theta\gamma_1 - \gamma_2\gamma_3 + \gamma_3[\sinh(\gamma_1 t) + \gamma_2\cosh(\gamma_1 t)]}{\cosh(\gamma_1 t) + \gamma_2\sinh(\gamma_1 t)} \\
C(t,T) &= -\frac{1}{2}\log[\cosh(\gamma_1 t) + \gamma_2\sinh(\gamma_1 t)] + \frac{1}{2}\kappa t + \frac{\kappa^2\theta^2\gamma_1^2 - \gamma_3^2}{2\sigma^2\gamma_1^3}(\frac{\sinh(\gamma_1 t)}{\cosh(\gamma_1 t) + \gamma_2\sinh(\gamma_1 t)} - \gamma_1 t) \\
&\quad + \frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3)\gamma_3}{\sigma^2\gamma_1^3}\frac{\cosh(\gamma_1 t) - 1}{\cosh(\gamma_1 t) + \gamma_2\sinh(\gamma_1 t)}
\end{aligned}$$

where $y = Y_0$ and $\hat{s}_1 = \frac{1}{2}u^2\bar{\rho}^2 + \frac{1}{2}iu(1-2\kappa\rho/\sigma)$, $\hat{s}_2 = iu\kappa\theta\rho\sigma^{-1}$, $\hat{s}_3 = \frac{1}{2}iu\rho\sigma^{-1}$. $\phi_t(u)$ is regular in a neighborhood of the origin, so by Theorem 7.1.1 in Lukacs [Luk70], $\phi_t(u)$ is also regular in the horizontal strip $\{u \in \mathbb{C} : p_-(t) < u < p_+(t)\}$, where

$$\begin{aligned}
p_+(t) &= \sup_{p\geq 1}\mathbb{E}(e^{pX_t}) < \infty\,, \\
p_-(t) &= \inf_{p\leq 0}\mathbb{E}(e^{pX_t}) < \infty
\end{aligned} \tag{3.3}$$

Note that $p_\pm(t)$ is not the same as $p_\pm$ as defined in the statement of the proposition, and we will show that $p_-(t) \le p_-$ and $p_+ \le p_+(t)$ (see discussion above (3.6)).

Looking at the expressions for B, C, D on page 12 in [SZ99], we see that $\phi_t(u)$ has a pole at $u = -ip$ if and only if

$$\cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t) \quad = \quad 0\,.$$

For $p \in (p_-, p_+)$ i.e. such that $\Gamma(p) > 0$, using that $\gamma_1 = \Gamma(p)$ and $-1/\gamma_2 = -\Gamma(p)/(\kappa - \rho p \sigma)$, this equation is satisfied if $t = t^*(p) = \frac{1}{\gamma_1}\tanh^{-1}(\frac{1}{\gamma_2}) = \frac{1}{\Gamma(p)}\tanh^{-1}(-\frac{\Gamma(p)}{\kappa - p\rho\sigma})$. But negative t-values are physically meaningless, so our preliminary analysis would indicate that

$$T^*(p) \quad = \quad \begin{cases} \frac{1}{\gamma_1}\tanh^{-1}(\frac{1}{\gamma_2}) \;=\; \frac{1}{\Gamma(p)}\tanh^{-1}(-\frac{\Gamma(p)}{\kappa - p\rho\sigma}) & (\kappa - p\rho\sigma < 0) \\ +\infty & (\kappa - p\rho\sigma \ge 0) \end{cases}$$

where $T^*(p) = \sup\{t : \mathbb{E}(e^{pX_t}) < \infty\}$ is the *moment explosion time*. We now first consider the case when $p > 1$. In this case, if $\rho \le 0$ then $\kappa - \rho p \sigma > 0$ and for $p \in (p_-, p_+)$ we have that $\Gamma(p) > 0$, so $T^*(p) = +\infty$. Otherwise, if $\rho > 0$, then $\kappa - p\rho\sigma < 0$ if $p > p^*$ where $p^* = \kappa/(\rho\sigma)$. However

$$p^* - p_+ \quad = \quad \frac{2\kappa - \rho\sigma - \rho\sqrt{4\kappa^2 - 4\kappa\rho\sigma + \sigma^2}}{2\rho\sigma\bar{\rho}^2}\,,$$

and using that

$$(2\kappa - \rho\sigma)^2 - \rho^2(4\kappa^2 - 4\kappa\rho\sigma + \sigma^2) \quad = \quad 4\kappa\bar{\rho}^2(\kappa - \rho\sigma) \quad > \quad 0$$

we see that $p^* > p_+$, so it turns out that $T^*(p) = \infty$ for *all* $p \in (1, p_+)$. An almost identical calculation shows that $T^*(p) = \infty$ for *all* $p \in (p_-, 0)$. Moreover, for $p \in [0, 1]$, from Jensen's inequality and the martingale property we have that $\mathbb{E}(S_t^p) \le S_0^p < \infty$ for all t. Thus we have shown that $T^*(p) = \infty$ for all $p \in (p_-, p_+)$, so the mgf of X_t is given by the analytic extension of ϕ to the imaginary axis for $p \in (p_-, p_+)$.

The expression for $C(.)$ in [SZ99] is given by

$$\begin{aligned} C(t, \hat{s}_1, \hat{s}_2, \hat{s}_3) \quad &= \quad -\frac{1}{2}\log[\cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t) + \frac{1}{2}\kappa t] \\ &+ \quad \frac{\kappa^2\theta^2\gamma_1^2 - \ \gamma_3^2}{2\sigma^2\gamma_1^3}(\frac{\sinh(\gamma_1 t)}{\cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t)} - \gamma_1 t) + \frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3)\gamma_3}{\sigma^2\gamma_1^3}(\frac{\cosh(\gamma_1 t) - 1}{\cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t)}) \end{aligned}$$

where $\gamma_1 = \sqrt{2\sigma^2\hat{s}_1 + \kappa^2}$, $\gamma_2 = (\kappa - 2\sigma^2\hat{s}_3)/\gamma_1$ and $\gamma_3 = \kappa^2\theta - \hat{s}_2\sigma^2$, and for $u = -ip$ and $p \in (p_-, p_+)$, using that $\gamma_1 = \Gamma(p) > 0$ and $\cosh(\gamma_1 t) \sim \sinh(\gamma_1 t) \sim e^{\gamma_1 t}$ as $t \to \infty$, we obtain the following large-time behavior for $C(t, \hat{s}_1, \hat{s}_2, \hat{s}_3)$:

$$C(t, \hat{s}_1) \quad \sim \quad \frac{1}{2}\big[\kappa + \frac{(p-1)p\theta^2\kappa^2}{\Gamma(p)^2} - \Gamma(p)\big] \quad = \quad t\big[V(p) + \frac{1}{2}p\rho\sigma\big] \qquad (t \to \infty)\,. \tag{3.4}$$

Letting $t \to \infty$ and using that $\coth(\gamma_1 t) \to 1$ as $t \to \infty$, we also find that

$$\begin{aligned} B(t, \hat{s}_1, \hat{s}_2, \hat{s}_3) \quad &\sim \quad -\frac{1}{\sigma^2\gamma_1}[\gamma_3 - \kappa\theta\gamma_1] \quad = \quad O(1) \qquad (t \to \infty)\,, \\ D(t, \hat{s}_1, \hat{s}_2, \hat{s}_3) \quad &\sim \quad \frac{1}{\sigma^2}(\kappa - \gamma_1) \quad = \quad O(1) \qquad (t \to \infty)\,, \end{aligned}$$

and thus constitute higher order terms as $t \to \infty$, which we can ignore at the order we are interested in. Thus we have

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}(e^{pX_t}) \quad = \quad V(p) \tag{3.5}$$

for $p \in (p_-, p_+)$. This means that for $p \in (p_-, p_+)$ and $t < \infty$ fixed, we have $\mathbb{E}(e^{pX_t}) < \infty$, so

$$\begin{aligned} p_-(t) \quad &\le \quad p_-\,, \\ p_+(t) \quad &\ge \quad p_+\,. \end{aligned} \tag{3.6}$$

We now consider $p \geq p_+$. To this end we fix a $q \in (1, p_+)$; then from the monotonicity of the L^p norm we have

$$(\mathbb{E}(e^{qX_t}))^{1/q} \quad \leq \quad (\mathbb{E}(e^{pX_t}))^{1/p}.$$

From this and (3.5) we obtain

$$V(q) \quad = \quad \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{qX_t})) \quad \leq \quad \liminf_{t\to\infty} \frac{1}{t} \log(\mathbb{E}(e^{pX_t}))^{q/p}.$$

But $\forall K > 0$, there exists a $q(K) < p_+$ such that $V(q) \geq K$. Thus for t sufficiently large we have

$$(e^{Kt})^{p/q(K)} \qquad \leq \qquad \mathbb{E}(e^{pX_t})$$

or

$$K \quad \leq \quad K \frac{p}{q(K)} \quad \leq \quad \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{pX_t}).$$

Thus letting $K \to \infty$ we see that $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{pX_t}) = +\infty$. A similar analysis shows that $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{pX_t}) = +\infty$ for $p \leq p_-$.

Differentiating $V(p)$ we obtain

$$V'(p) \quad = \quad \frac{(2p-1)\theta^2\kappa^2 + 2(1-p)p\theta^2\kappa^2\Gamma'(p)}{\Gamma(p)^2} - \frac{1}{2}(\rho\sigma + \Gamma'(p))$$

and

$$\Gamma'(p) \quad = \quad \frac{\sigma(-2\kappa\rho + (1 + 2p(-1+\rho^2))\sigma)}{2\Gamma(p)}.$$

Nothing that $\Gamma(p_\pm) = 0$, we see that $V(p)$ and $|V'(p)| \to +\infty$ as $p \to p_\pm$ so V is essentially smooth, and V is lower semicontinuous. $V_t(p) = \mathbb{E}(e^{pX_t})$ satisfies Assumption 2.3.2 in [DZ98] as $t \to \infty$, so by Lemma 2.3.9 in [DZ98] V is also convex, so from the Gärtner-Ellis Theorem (see Theorem 2.3.6 in [DZ98]) X_t/t satisfies the LDP with good convex rate function $I(x)$.

We also have the upper bound

$$I(x) \quad \leq \quad p_+ x \vee p_- x - V_{\min} \quad < \quad \infty$$

where $V_{\min} = \inf_{p\in(p_-,p_+)} V(p) > -\infty$. But a convex function is continuous on the interior of its domain, so I is continuous. Finally, from elementary calculations we find that the unique minimum of I occurs at $x^* = (I')^{-1}(0) = V'(0)$. □

4. Call options and implied volatility

Let $\mathbb{P}^*(A) = \frac{1}{S_0}\mathbb{E}(S_t 1_A)$ for $A \in \mathcal{F}_t$ denote the *Share measure* ($\mathbb{P}^*$ is a probability measure because S_t is a martingale by Proposition 3.1). From Girsanov's theorem, it is easily shown that

$$\begin{cases} d(-X_t) &= -\frac{1}{2}Y_t^2 dt - Y_t dW_t^{*1}, \\ dY_t &= [\kappa(\theta - Y_t) + \rho\sigma Y_t]\, dt + \sigma dW_t^{*2} \\ &= \bar{\kappa}(\bar{\theta} - Y_t)dt + \sigma dW_t^{*2}, \end{cases} \tag{4.1}$$

where $\bar{\kappa} = \kappa - \rho\sigma$, $\bar{\theta} = \kappa\theta/(\kappa - \rho\sigma)$ and $dW_t^{*1} dW_t^{*2} = \rho dt$ are independent $\mathbb{P}^*$-Brownian motions.

Assumption 4.1. *From here on we further assume that $\bar{\kappa} = \kappa - \rho\sigma > 0$, which ensures that Y_t is mean-reverting under $\mathbb{P}^*$.*

From (4.1), we have the following trivial corollary of Proposition 3.2.

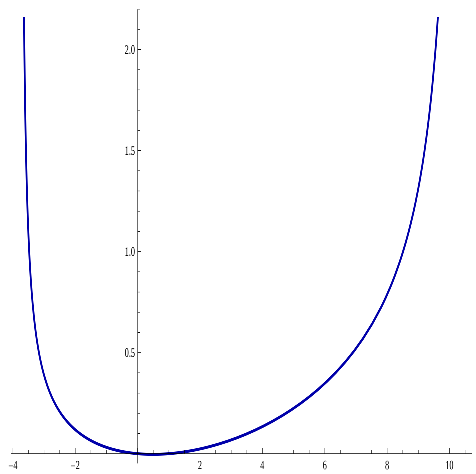

FIG 1. *Here we have plotted $V(p)$ for $\kappa = 1.15, \theta = 0.1, \sigma = 0.2, \rho = -0.4$.*

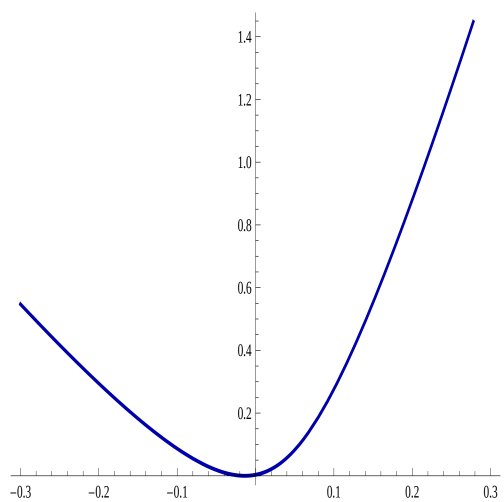

FIG 2. *Here we have plotted the rate function $I(x)$ for the same parameter values as above.*

Corollary 4.2. *For $\kappa > \rho\sigma$, $-X_t/t$ satisfies the LDP under $\mathbb{P}^*$ as $t \to \infty$ with a good convex continuous rate function $I_S(x)$ given by the Fenchel-Legendre transform of*

$$V_S(p) \quad = \quad V(p; \bar{\kappa}, \bar{\theta}, \sigma, -\rho)$$

and I_S is continuous and attains its minimum value uniquely at $-x^+ = (V_S)'(0) = -\frac{1}{2}(\bar{\theta}^2 + \sigma^2/2\bar{\kappa})$.

By Corollary 4.2 and the continuity of the rate function I_S, we obtain the following corollary, which will be used to characterize the large-time behaviour of call option prices.

Corollary 4.3.

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}^*(X_t > xt) \quad &= \quad -I_S(x) \qquad\qquad (x > x_+)\,, \\
\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}^*(X_t < xt) \quad &= \quad -I_S(x) \qquad\qquad (x < x_+)\,.
\end{aligned}$$

Recall that the payoff of a European call option with strike K is $\mathbb{E}(S_t - K)^+$, and the payoff of a European put option with strike K is $\mathbb{E}(K - S_t)^+$.

Corollary 4.4. *We have the following large-time asymptotic behaviour for put/call options in the large-time, large log-moneyness regime:*

$$\begin{aligned}
-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(S_t - S_0 e^{xt})^+ \quad &= \quad I_S(x) \qquad\qquad (x \geq x_+)\,, \\
-\lim_{t\to\infty} \frac{1}{t} \log[S_0 - \mathbb{E}(S_t - S_0 e^{xt})^+] \quad &= \quad I_S(x) \qquad\qquad (x^* \leq x \leq x_+)\,, \\
-\lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}(S_0 e^{xt} - S_t)^+) \quad &= \quad I_S(x) \qquad\qquad (x \leq x^*)
\end{aligned}$$

Proof. We first assume $x > x_+$, and recall that $I_S(x)$ is non-decreasing for $x > x_+$. From Corollary 4.3, we know that for all $\varepsilon > 0$ there exists a $t^* = t^*(\varepsilon)$ such that for all $t > t^*$ we have

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{xt})^+ \quad = \quad \mathbb{P}^*(X_t > xt) - e^{xt}\mathbb{P}(X_t > xt) \quad \leq \quad \mathbb{P}^*(X_t > xt) \quad \leq \quad e^{-(I_S(x)-\varepsilon)/t}$$

which gives the upper bound for the call price. For the lower bound we have

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{xt})^+ \quad = \quad \mathbb{E}^{\mathbb{P}^*}(1 - e^{xt}e^{-X_t})^+ \quad = \quad e^{xt}\mathbb{E}^{\mathbb{P}^*}(e^{-xt} - e^{-X_t})^+\,. \tag{4.2}$$

Observe that for any $\delta > 0$,

$$\mathbb{E}^{\mathbb{P}^*}(e^{-xt} - e^{-X_t})^+ \geq \mathbb{E}^{\mathbb{P}^*}[(e^{-xt} - e^{-X_t})^+ 1_{\{-X_t < -xt-\delta\}}] \geq (e^{-xt} - e^{-xt-\delta})\mathbb{P}^*(-X_t < -xt - \delta)\,.$$

Combining this with (4.2) we have

$$\begin{aligned}
\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{xt})^+ \quad &\geq \quad e^{xt}(e^{-xt} - e^{-xt-\delta})\mathbb{P}^*(-X_t < -xt - \delta) \\
&= \quad (1 - e^{-\delta})\mathbb{P}^*(-X_t < -xt - \delta) \\
&= \quad (1 - e^{-\delta})\mathbb{P}^*(X_t/t > x + \delta/t) \\
&\geq \quad (1 - e^{-\delta})\mathbb{P}^*(X_t/t > x + \delta).
\end{aligned}$$

Using Corollary 4.3 we get

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(S_t - S_0 e^{xt})^+ \geq I_S(x + \delta).$$

This holds for all $\delta > 0$, so taking $\lim_{\delta\to 0}$ and by the continuity of $I_S(x)$ we obtain the first result that $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(S_t - S_0 e^{xt})^+ = I_S(x)$. The other cases follow similarly.

□

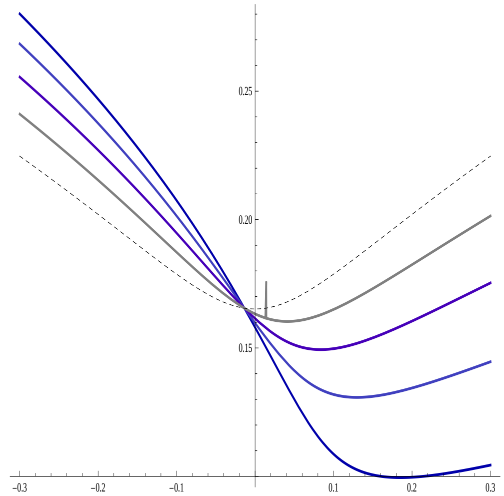

FIG 3. *Here we have plotted the asymptotic implied volatility $\hat{\sigma}(x)$ for $\kappa = 1.15, \theta = 0.1, \sigma = 0.2$ and $\rho = -.8, -.6, -.4, -.2$ and 0 (in blue, light blue, purple, grey and black dashed respectively).*

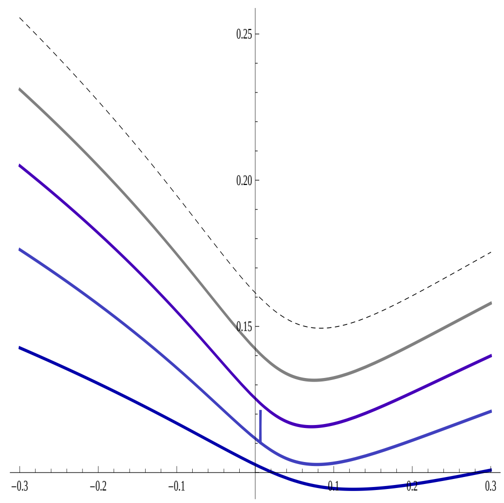

FIG 4. *Here we have plotted the asymptotic implied volatility $\hat{\sigma}(x)$ for $\kappa = 1.15, \theta = 0.1, \rho = -0.4$ and $\sigma = .04, .08, .12, .16$ and .2 (in blue, light blue, purple, grey and black dashed respectively).*

4.1. Implied volatility

Using the same proofs as in Corollary 1.7 and Corollary 2.17 in [FJ11] for the Heston model (or Theorem 14 in [JKRM13] for a general affine model), we have the following asymptotic behaviour in the large-time, large log-moneyness regime, where $\hat{\sigma}_t(xt)$ is the implied volatility of a put/call option with strike $S_0 e^{xt}$ for the correlated Stein-Stein model:

$$\hat{\sigma}_\infty(x)^2 \;=\; \lim_{t\to\infty} \hat{\sigma}_t^2(xt) \;=\; \begin{cases} 2(2I(x) - x - 2\sqrt{I(x)^2 - I(x)x}) & (x \notin [x^*, x_+]) \\ 2(2I(x) - x + 2\sqrt{I(x)^2 - I(x)x}) & (x \in (x^*, x_+))\,. \end{cases}$$

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