Convergence to Lévy marginals for a re-scaled rough Heston model with jumps

Alessandro Bondi* Martin Forde†

Abstract

We prove weak convergence of the log-price marginals for a re-scaled rough Heston model to the marginals of a Normal Inverse Gaussian (NIG) Lévy process. This establishes a direct connection between the two classes of processes, without using a Markovian approximation as in the reversionary Heston model of [AC24] (where the true Hurst index $H = \frac{1}{2}$), or limits of hyper-rough models as in [AAR25] (where $H \searrow -\frac{1}{2}$). We then extend these results to the case of the rough Hawkes–Heston model introduced in [BPS24], where the variance process has an additional self-exciting jump component, and the log-price process has a leverage/feedback term driven by the same component which generates negative jumps/skew. We also provide an explicit characterization of the limiting time-changed Lévy process, which extends the NIG class.

1 Introduction

For a given $\alpha \in (\frac{1}{2}, 1)$, we consider processes $X = (X_t)_{t\geq 0}$ and $V = (V_t)_{t\geq 0}$ satisfying

$$dX_t = -\left(\frac{1}{2} + \int_{\mathbb{R}_+} \left(e^{-\Lambda z} - 1 + \Lambda z\right)\nu(dz)\right) V_t\, dt + \sqrt{V_t}\, dB_t - \Lambda\, d\tilde{J}_t, \qquad X_0 = 0, \tag{1}$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\lambda(\theta - V_s)\, ds + \sigma\sqrt{V_s}\, dW_s + d\tilde{J}_s\Big). \tag{2}$$

In this model, W, B are two standard Brownian motions with $\langle B, W\rangle_t = \rho t$ for all $t \geq 0$, and $\tilde{J} = (\tilde{J}_t)_{t\geq 0}$ is the purely discontinuous local martingale defined by $\tilde{J}_t = \int_0^t \int_{\mathbb{R}_+} x\big(N(dx, ds) - V_s\nu(dx)\, ds\big)$, $t \geq 0$. Here, $N(dx, dt)$ is an integer-valued random measure with compensator $V_t\nu(dx)\, dt$, and ν is a non-negative measure with support contained in $(0, \infty)$ and finite second moment, namely $\nu((-\infty, 0]) = 0$, $\int_{\mathbb{R}_+} x^2\, \nu(dx) < \infty$. Since the compensator of N is proportional to the solution process V, the model exhibits a self-exciting jump mechanism.

Equation (2) is an affine stochastic Volterra equation (SVE) of convolution type with jumps, whose weak well-posedness is established in [AJ21, BLP24]. Since the fractional kernel $t \mapsto \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$ is singular and introduces a dependence on the past, V is neither a semimartingale nor Markovian. On the other hand, we notice that the process X in (1) is fully determined by V, the measure ν, and the leverage parameter $\Lambda > 0$.

The pair (X, V) constitutes the rough Hawkes–Heston stochastic volatility model from [BPS24], where X represents the log-price process and V the variance process. The model can be viewed as an extension of the rough Heston model of [JR16] incorporating the jump component $\tilde{J}$. This term admits only positive jumps and drives both the variance process V, see (2), and the log-price process X in (1). The corresponding jumps in X are negative due to Λ. As in the continuous case $\nu = 0$, the model (1)-(2) belongs to the class of affine processes. [BLP24] derive the conditional Fourier–Laplace transform of X in terms of a nonlinear Volterra integral equations (VIE), so we can price options on X via Fourier inversion.

In this paper, we leverage this formula to investigate weak scaling limits of the time-t marginal distribution of X, for every $t > 0$. More specifically, we show that the time-t marginal of a suitably rescaled version of X in (1), indexed by a parameter ε and denoted by X^ε, converges weakly to the corresponding marginal of a time-changed Lévy process with drift belonging to a class that extends the *Normal Inverse Gaussian* (NIG) family. For this we use an asymptotic result from [FGS21], which we exploit to characterize the limit of the mgf of X^ε. The weak convergence is then established by identifying a drifted time-changed Lévy process whose mgf coincides with this limiting expression.

This work fits into a recent strand of the literature investigating connections between Volterra–Heston-type models and Lévy processes. In the continuous setting, corresponding to $\nu = 0$, the first paper to identify such a connection was [AC24]. There, the authors consider dynamics driven by a shifted power-law kernel $t \mapsto (t + \varepsilon)^{H-\frac{1}{2}}$ and, through a suitable Markovian approximation called the reversionary Heston model, show that for $H = -\frac{1}{2}$ the log-price process [resp., the integrated variance] converges weakly to an NIG process [resp., an *Inverse Gaussian* (IG) process] as the shift parameter $\varepsilon \to 0$. This approach therefore establishes an indirect connection based on proxy models with Hurst index $H = \frac{1}{2}$, since the dynamics remain driven by a standard Brownian motion. A direct connection is established in [AAR25], where IG processes arise as weak limits of the integrated variance in so-called hyper-rough Heston models. These models are driven by power-law kernels that are only locally integrable, namely $t \mapsto t^{H-\frac{1}{2}}$, with Hurst index $H = \alpha - \frac{1}{2} \in (-\frac{1}{2}, 0)$. In this regime, IG processes

*Department of AI, Data and Decision Sciences, Luiss University, Rome. abondi@luiss.it
†Department of Mathematics, King's College London. https://martinforde.github.io

emerge in the limit $H \searrow -\frac{1}{2}$. We also refer the reader to the recent work [AAS26], which studies nonlinear processes – called *Volterra clocks* – and characterizes their weak limits as first-passage times of Brownian motion to curved boundaries.
In the jump setting, namely when $\nu \neq 0$, weak scaling limits of the integrated variance process $t \mapsto \int_0^t V_s ds$, where V is given by (2), are investigated in [BF26]. There, the limiting Lévy process is characterized as first-passage time to negative barriers of a spectrally positive Lévy process.

We now review some interesting (related) works with similar dynamics to (1)-(2), albeit with different modelling objectives. In the no-jumps case, Sections 3.1-3.2 in [BJ26] prove that the V process in (2) for the rough Heston (rHeston) model has a positive atom at zero. Specifically, using the solution $y_\lambda(\cdot)$ to the VIE associated with $\mathbb{E}[e^{-\lambda V_T}]$, they show that $y_\lambda(t) \leq At^{-\alpha}$ for all $\lambda > 0$ and $t > 0$, obtained by combining a fractional maximum principle and that $At^{-\alpha}$ is a strict supersolution of the VIE. This yields a lower bound for $\mathbb{E}[e^{-\lambda V_T}]$ (uniformly in λ) and letting $\lambda \to \infty$ yields that $\mathbb{P}(V_T = 0) > 0$. The result shows that, unlike the classical CIR/Heston case or more general Volterra dynamics driven by nonsingular kernels discussed in [BP26] (see, in particular, Section 3.1 in [BP26]), no Feller-type inequality can make the zero boundary inaccessible for $H < \frac{1}{2}$. The same result is essentially shown in [FGW26] (where the authors also give a lower bound for $\mathbb{P}(V_T = 0)$ on page 19), and in Section 1.5 they prove the surprising result that an rHeston model with constant drift $\mu \neq r$ under $\mathbb{P}$ admits no equivalent local martingale measure. In particular, equivalent martingale measures exist only under very restrictive assumptions on the drift under $\mathbb{P}$. Indeed, if $L_t = \int_0^t \mathbf{1}_{\{V_s=0\}}\, ds$, then $\mathbb{E}[L_t] = \int_0^t \mathbb{P}(V_s = 0)\, ds > 0$, so $\mathbb{P}(L_t > 0) > 0$. Moreover, since V is continuous, the zero set $\mathcal{Z} := \{s \in [0,t] : V_s = 0\}$ is closed, and L_t is precisely its Lebesgue measure. However, one can also easily show that V cannot be identically zero on any non-empty interval (see also Lemma 4.9 in [JP22]), so $\mathcal{Z}$ contains no interval and hence has an empty interior. Thus, $\mathbb{P}\big(\mathcal{Z}$ is closed, has positive Lebesgue measure, and has empty interior$\big) > 0$, i.e., the zero set exhibits fat-Cantor-like behaviour with positive probability. In Theorem 1.5 in [FGW26], they also derive an exact closed-form expression for $\mathbb{P}(V_T = 0)$ when the mean reversion is zero.

[JR20] and [MORS26] model the *signed order flow* F_t with Hawkes/Rough Heston-type processes as discussed above, and assume the price process is *endogenously* determined as $P_t = \kappa \lim_{u\to\infty} \mathbb{E}[F_u|\mathcal{F}_t]$ (*). If we assume instead that P is a given continuous semimartingale, and $F_t = \int_0^t f(t-s)dP_s$, with $f(t) = ct^\gamma$ for $c > 0$, $\gamma \in (0, \frac{1}{2})$, then we have the *inversion formula*: $P_t = P_0 + \frac{d}{dt}\int_0^t h(t-s)F_s ds$, where $h * f' \equiv 1$ (which is satisfied if $h(t) = \frac{\sin(\pi\gamma)}{c\gamma\pi} t^{-\gamma}$). If P is also $\frac{1}{2} - \varepsilon$-Hölder-continuous for all $\varepsilon > 0$, by the standard mapping property of the operator $F = c\Gamma(\gamma+1)I^\gamma(P - P_0) \in C^{\gamma+\frac{1}{2}-\varepsilon}(0,T)$ [1], so F is smoother than Brownian motion (consistent with empirical findings in [MORS26]). Since $\gamma + \frac{1}{2} - \varepsilon > \gamma$ for $\varepsilon \ll 1$, Young integration by parts and the standard parameter-differentiation rule for Young convolutions imply that P can be rewritten as the Young integral $P_t - P_0 = \int_0^t h(t-s)dF_s := \lim_{\delta\downarrow 0}\int_0^{t-\delta} h(t-s)dF_s$ (i.e. the *propagator equation*). This leads to *concave price impact* and the empirically well-documented square-root impact law in the limit as $\gamma \uparrow \frac{1}{2}$ for additional exogenous trades with constant trading speed. This approach does not require the strong assumption (*) above, or that P is driven by a hyper-rough Heston-type process as in [MORS26], and allows us to choose H and γ independently if P follows a rough volatility model with Hurst parameter H.

The quadratic rough Heston (qRHeston) model introduced in [GJR20] (see also [BNPRS26]) is a non-affine rough model, for which Monte Carlo approximations have generally been more successful than the rHeston model in jointly fitting SPX and VIX smiles[2]. The diffusion coefficient $\sqrt{a(Z_t - b)^2 + c}$ in the SVE for the process Z in the model is Lipschitz for $c > 0$, so we can appeal to the strong existence/uniqueness results in Theorem 3.3 and Lemma 3.1 in [ALP19]. However, it is unclear whether the price process S is a true martingale, since the drift for the Z process explodes quadratically when Z is large under the share measure $\mathbb{P}^*$, and for $H = \frac{1}{2}$, one can easily verify that S is a strict local martingale via the usual Feller boundary classification for V under $\mathbb{P}^*$ [3]. To circumvent this, we can just modify the drift or volatility coefficient of the SVE for Z (but that prevents us from using the exact VIX sampling formula in Section 6.2 in [Rom22]), or we can use e.g. the alternate generalized CEV Volterra model from Section 6.3 in [Rom22], for which this is not an issue. The price process S will of course be a martingale for a standard log-Euler Monte Carlo scheme.

This paper is organized as follows. In Section 2, we consider a rescaled version of (1)-(2) in the case $\nu = 0$, corresponding to the rough Heston model, and prove the convergence of the log-stock price marginals to those of an NIG process, see Theorem 2.1. This result is of independent interest, as it covers a relevant example in the literature, and serves as a motivation for the more general procedure developed in the sequel. In Section 3, we extend this approach to the jump setting $\nu \neq 0$, corresponding to the rough Hawkes–Heston model studied in [BLP24]. More precisely, Proposition 3.1 establishes the limiting behavior of the mgf of a suitable rescaling of the log-price process in (1). Theorem 3.2 in Subsection 3.1 then shows that the limiting expression corresponds to the mgf of a drifted time-changed Lévy process belonging to a class that extends the NIG family.

2 Asymptotics for the log stock price for a re-scaled rough Heston model

We consider processes defined on (possibly different) probability spaces $(\Omega, \mathcal{F}, \mathbb{Q})$, each equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Indeed, since existence (and uniqueness) for the model (1)-(2) is

[1] I^γ denotes the γ-th order fractional integral.
[2] see e.g. [BG25] and notes/code on the second author's website.
[3] proof available on request

established only in the weak sense (see [ALP19] for the continuous case $\nu \equiv 0$, and [AJ21], [BLP24] for the jump case), every scaling of this model is defined on possibly different stochastic bases. We start by recalling the definition of Normal Inverse Gaussian distribution, which appears in the scaling limit in distribution established in Theorem 2.1.

Definition 2.1 A random variable X is said to have a *Normal Inverse Gaussian (NIG)* distribution with parameters $\hat{\alpha}, \delta > 0$, $\mu \in \mathbb{R}$ and $\beta \in (-\hat{\alpha}, \hat{\alpha})$, denoted by $X \sim NIG(\hat{\alpha}, \beta, \delta, \mu)$, if its probability density function is

$$f_X(x) = \frac{\hat{\alpha}\delta}{\pi} \frac{K_1\big(\hat{\alpha}\sqrt{\delta^2 + (x-\mu)^2}\big)}{\sqrt{\delta^2 + (x-\mu)^2}} \exp\Big\{\delta\sqrt{\hat{\alpha}^2 - \beta^2} + \beta(x-\mu)\Big\}, \quad x \in \mathbb{R},$$

where K_1 denotes the modified Bessel function of the second kind.

The moment generating function of $X \sim NIG(\hat{\alpha}, \beta, \delta, \mu)$ is given by

$$\mathbb{E}\big[e^{pX}\big] = \exp\Big\{\mu p + \delta\gamma - \delta\sqrt{\gamma^2 - p(p+2\beta)}\Big\}, \quad p \in (-\hat{\alpha} - \beta, \hat{\alpha} - \beta),$$
$$\text{where } \gamma = \sqrt{\hat{\alpha}^2 - \beta^2}. \tag{3}$$

An NIG process with parameters $(\hat{\alpha}, \beta, \delta, \mu)$ is a (càdlàg) Lévy process $L = (L_t)_{t\geq 0}$ where $L_1 \sim NIG(\hat{\alpha}, \beta, \delta, \mu)$. By Example 1.3.32 in [Appl09], an $NIG(\hat{\alpha}, \beta, \delta, \mu)$ process $L = (L_t)_{t\geq 0}$ can be constructed with a subordination approach as follows, recalling $\gamma = \sqrt{\hat{\alpha}^2 - \beta^2}$ defined in (3):

$$L_t = \mu t + \beta\tau_t + W^{(1)}_{\tau_t}, \quad \text{with} \quad \tau_t = \inf\{s \geq 0 \,:\, \gamma s + W^{(2)}_s = \delta t\}, \tag{4}$$

where $W^{(1)}$ and $W^{(2)}$ are two independent standard Brownian motions.

In the next theorem, Theorem 2.1, we consider a rescaled version of the dynamics in (1)-(2) with $\nu \equiv 0$ (that is, the rough Heston model), and prove that the time-t marginals of the log-price process converge to those of an NIG Lévy process independent of the initial variance V_0 and the exponent α in the fractional kernel. The proof relies on the affine structure of the model, which yields a semi-closed expression for the moment generating function of log prices in terms of deterministic Riccati–Volterra equations. In Figure 2, we visualize the weak convergence result in Theorem 2.1 at the level of implied volatility (IV) smiles.

Theorem 2.1 *Consider the rescaled rough Heston model*

$$dX^\varepsilon_t = -\frac{1}{2}V^\varepsilon_t dt + \sqrt{V^\varepsilon_t}dB_t, \qquad X^\varepsilon_0 = 0, \tag{5}$$
$$V^\varepsilon_t = V_0 + \frac{1}{\varepsilon\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\lambda(\theta - V^\varepsilon_s)ds + \sigma\sqrt{V^\varepsilon_s}dW_s), \qquad V_0 \geq 0, \tag{6}$$

where B and W are two Brownian motions with correlation $\langle B, W\rangle_t = \rho t$, $\rho \in (-1, 0]$, $\alpha \in (\frac{1}{2}, 1)$ and $\lambda, \theta, \sigma > 0$. Denote by $X = (X_t)_{t\geq 0}$ an $NIG(\hat{\alpha}, \beta, \delta, \mu)$ Lévy process with

$$\beta = \frac{1}{\bar{\rho}^2}\Big(\frac{\lambda\rho}{\sigma} - \frac{1}{2}\Big), \qquad \hat{\alpha} = \sqrt{\beta^2 + \gamma^2}, \text{ with } \gamma = \frac{\lambda}{\sigma\bar{\rho}}, \qquad \delta = \lambda\theta\frac{\bar{\rho}}{\sigma}, \qquad \mu = -\lambda\theta\frac{\rho}{\sigma}, \tag{7}$$

where $\bar{\rho} = \sqrt{1-\rho^2} > 0$. Then, for all $t > 0$ fixed, X^ε_t tends weakly to X_t as $\varepsilon \to 0$.

Remark 2.1 *As already mentioned before Theorem 2.1, the limiting NIG process X does not depend on V_0 or the Hurst index $H = \alpha - \frac{1}{2}$, see the parameters in* (7).

Proof. Denote by $I^\alpha(f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds$ the α-th order fractional integral of a function f. By, e.g. Section 7 in [ALP19], for $p \in (0,1)$

$$\mathbb{E}[e^{pX^\varepsilon_t}] = \exp\Big\{V_0 I^{1-\alpha}\phi_\varepsilon(t) + \frac{1}{\varepsilon}\lambda\theta I^1\phi_\varepsilon(t)\Big\}, \tag{8}$$

where ϕ_ε is the unique continuous $\mathbb{R}_-$−valued solution of the Riccati–Volterra integral equation (VIE):

$$\phi_\varepsilon(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\frac{1}{2}(p^2 - p) + \frac{1}{\varepsilon}(\rho p\sigma - \lambda)\phi_\varepsilon(s) + \frac{1}{\varepsilon^2}\frac{1}{2}\sigma^2\phi_\varepsilon(s)^2\Big)ds, \quad t \geq 0.$$

Now define the map ψ_q by $\psi_q(t) = \frac{1}{\varepsilon}\phi_\varepsilon(\frac{t}{\varepsilon^q})$, for some $q \in \mathbb{R}$. Then, denoting by

$$F(w) = \frac{1}{2}(p^2 - p) + (\rho p\sigma - \lambda)w + \frac{1}{2}\sigma^2 w^2, \quad w \in \mathbb{R},$$

the change of variables $\varepsilon^q s = u$ shows that

$$\varepsilon\psi_q(t) = \frac{1}{\Gamma(\alpha)}\int_0^{t/\varepsilon^q}\Big(\frac{t}{\varepsilon^q} - s\Big)^{\alpha-1}\Big(\frac{1}{2}(p^2 - p) + (\rho p\sigma - \lambda)\psi_q(\varepsilon^q s) + \frac{1}{2}\sigma^2\psi_q(\varepsilon^q s)^2\Big)ds$$
$$= \frac{\varepsilon^{-q\alpha}}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}F(\psi_q(u))du. \tag{9}$$

If we let $q = -\frac{1}{\alpha}$, the VIE (9) is independent of ε, and so is its unique continuous solution $\psi = \psi_{-\alpha^{-1}}$. An application of Lemma 4.5 in [FGS21] (see also the proof of Theorem 2.1 in [BF26]) entails that

$$\lim_{t\to\infty} \psi(t) \quad = \quad U_1(p) \quad := \quad \frac{1}{\sigma^2}\Big[\lambda - p\sigma\rho - \sqrt{\lambda^2 - 2\lambda\rho\sigma p + \sigma^2 p(1-p\bar{\rho}^2)}\Big],$$

where $U_1(p)$ is the smallest (equivalently, the negative) root of F, and we recall that $\bar{\rho} = \sqrt{1-\rho^2} > 0$. Since, by definition, $\phi_\varepsilon(t) = \varepsilon\psi(\varepsilon^{-1/\alpha}t)$, for every $t > 0$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\phi_\varepsilon(t) \quad = \quad \lim_{s\to\infty}\psi(s) \quad = \quad U_1(p).$$

As a result, the nonpositive continuous function $\frac{1}{\varepsilon}\phi_\varepsilon(\cdot)$ is bounded on $\mathbb{R}_+$, uniformly in ε. Therefore, for the exponent in (8), by the dominated convergence theorem

$$\begin{aligned} V_0 I^{1-\alpha}\phi_\varepsilon(t) + \frac{\lambda\theta}{\varepsilon}I^1\phi_\varepsilon(t) \quad &= \quad \frac{V_0}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\phi_\varepsilon(s)ds + \lambda\theta\int_0^t \frac{1}{\varepsilon}\phi_\varepsilon(s)ds \\ &\underset{\varepsilon\to 0}{\longrightarrow} \quad 0 + \lambda\theta U_1(p)t. \end{aligned} \tag{10}$$

By (3), explicit computations show that $\lambda\theta U_1(p)t$ is the log moment generating function of an NIG Lévy process at time t with parameters $(\hat{\alpha}, \beta, \delta, \mu)$ specified as in (7). Then, since p varies in $(0, 1)$, from, e.g., Problem 30.4 on Page 573 in [Bill86], X_t^ε tends weakly to the time-t marginal law of an $NIG(\hat{\alpha}, \beta, \delta, \mu)$ process, completing the proof. ■

3 Adding jumps: The Rough Hawkes–Heston model

We now generalize the processes $(X^\varepsilon, V^\varepsilon)$ in (5)-(6) by introducing jumps. In particular, choosing again $\alpha \in (\frac{1}{2}, 1)$, for any $\varepsilon > 0$ we consider

$$V_t^\varepsilon \quad = \quad V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\frac{1}{\varepsilon}\lambda(\theta - V_s^\varepsilon)ds + \frac{1}{\varepsilon}\sigma\sqrt{V_s^\varepsilon}dW_s + \frac{1}{\varepsilon}d\tilde{J}_s^\varepsilon\Big), \tag{11}$$

where $\tilde{J}_t^\varepsilon = \int_0^t\int_{\mathbb{R}_+} x(N^\varepsilon(dx, ds) - V_s^\varepsilon\nu(dx)ds)$ and $N^\varepsilon(dx, dt)$ is an integer-valued random measure with compensator $V_t^\varepsilon\nu(dx)dt$. As in [AJ21, BF26, BPS24], see also [BLP24], we assume ν only has positive support with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}_+}|x|^2\nu(dx) < \infty$. Consequently, $\tilde{J}^\varepsilon$ has positive-only jumps. With the change of variables $x = \varepsilon z$, we have

$$\frac{1}{\varepsilon}\tilde{J}_t^\varepsilon = \frac{1}{\varepsilon}\int_0^t\int_0^\infty x(N^\varepsilon(dx, ds) - V_s^\varepsilon\nu(dx)ds) = \int_0^t\int_0^\infty z(N^\varepsilon(\varepsilon\cdot dz, ds) - V_s^\varepsilon\nu(\varepsilon\cdot dz)ds).$$

As for the log stock price process $X^\varepsilon = (X_t^\varepsilon)_{t\geq 0}$, we model it with the dynamics

$$\begin{aligned} dX_t^\varepsilon &= -\Big(\frac{1}{2} + \int_{\mathbb{R}_+}(e^{-\Lambda z} - 1 + \Lambda z)\nu(dz)\Big)V_t^\varepsilon dt + \sqrt{V_t^\varepsilon}dB_t - \Lambda d\tilde{J}_t^\varepsilon \\ &= -\Big(\frac{1}{2} + \int_{\mathbb{R}_+}(e^{-\varepsilon\Lambda z} - 1 + \varepsilon\Lambda z)\nu(\varepsilon\cdot dz)\Big)V_t^\varepsilon dt + \sqrt{V_t^\varepsilon}dB_t \\ &\quad - \varepsilon\Lambda\int_{\mathbb{R}_+} z\Big(N^\varepsilon(\varepsilon\cdot dz, dt) - V_t^\varepsilon\nu(\varepsilon\cdot dz)dt\Big), \end{aligned} \tag{12}$$

for a given leverage parameter $\Lambda > 0$. The couple $(X^\varepsilon, V^\varepsilon)$ in (11)-(12) constitutes a rescaled rough Hawkes–Heston model (see (1)-(2) in Section 1), introduced and applied to real market data in [BPS24]. Note that, in the absence of jumps (that is, $\nu \equiv 0$), the previous dynamics are continuous and reduce to (5)-(6).

The model in (11)-(12) is affine and [BPS24] provides a semi-closed formula for the moment generating function of X^ε. More specifically, define the map

$$\mathcal{V}_\varepsilon(p, v) = \int_{\mathbb{R}_+}\Big(e^{(v-\varepsilon\Lambda p)z} - p(e^{-\varepsilon\Lambda z} - 1) - 1 - vz\Big)\nu(\varepsilon\cdot dz), \quad p\in[0,1],\, v\leq 0. \tag{13}$$

We note that $v \leq 0$ and $\Lambda p \geq 0$ ensure the convergence of the above integral, hence the finiteness of $\mathcal{V}_\varepsilon(p, v)$. Indeed, since $e^x - 1 - x \leq x^2$ for $x \leq 0$, by the hypotheses on ν,

$$\begin{aligned} |\mathcal{V}_\varepsilon(p, v)| &\leq \int_{\mathbb{R}_+}\Big(e^{(v-\varepsilon\Lambda p)z} - 1 - (v - \varepsilon\Lambda p)z\Big)\nu(\varepsilon\cdot dz) + p\int_{\mathbb{R}_+}\Big(e^{-\varepsilon\Lambda z} - 1 + \varepsilon\Lambda z\Big)\nu(\varepsilon\cdot dz) \\ &\leq \Big((v - \varepsilon\Lambda p)^2 + p\varepsilon^2\Lambda^2\Big)\int_{\mathbb{R}_+}|z|^2\nu(\varepsilon\cdot dz) < \infty, \quad p\in[0,1],\, v\leq 0. \end{aligned}$$

We also observe that the change of variables $x = \varepsilon z$ yields

$$\mathcal{V}_\varepsilon(p, \varepsilon v) = \mathcal{V}_1(p, v), \quad p\in[0,1],\, v\leq 0, \tag{14}$$

where $V_1(p,v)$ is given in (13) with $\varepsilon = 1$. Consider now the function

$$F_\varepsilon(p,v) = \frac{1}{2}(p^2-p) + \frac{1}{\varepsilon}(\rho p\sigma - \lambda)v + \frac{1}{2}\frac{1}{\varepsilon^2}\sigma^2 v^2 + V_\varepsilon(p,v), \quad p\in[0,1],\ v\le 0. \tag{15}$$

Then, from Theorem 3 in [BPS24], the moment generating function of X_t^ε is

$$\mathbb{E}\Big[e^{pX_t^\varepsilon}\Big] = \exp\Big\{V_0 I^{1-\alpha}\phi_\varepsilon(t) + \frac{1}{\varepsilon}\lambda\theta I^1\phi_\varepsilon(t)\Big\}, \quad p\in(0,1), \quad t\ge 0, \tag{16}$$

where $\phi_\varepsilon\colon \mathbb{R}_+ \to \mathbb{R}_-$ is the unique (nonpositive) continuous solution of the VIE

$$\phi_\varepsilon(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F_\varepsilon(p,\phi_\varepsilon(s))ds, \quad t\ge 0. \tag{17}$$

In the next proposition, we study the limit of (16) as $\varepsilon\to 0$. This result is crucial for describing the limit (in distribution) of the marginal laws of X^ε in Subsection 3.1.

Proposition 3.1 *For every $p\in(0,1)$,*

$$\textit{define } U_J(p) \textit{ as the unique negative root of the map } F_1(p,\cdot) \textit{ on } \mathbb{R}_-, \tag{18}$$

where $F_1(p,\cdot)$ is given in (15) with $\varepsilon = 1$. Then, for every $t>0$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[e^{pX_t^\varepsilon}\Big] = \exp\Big\{\lambda\theta U_J(p)t\Big\}. \tag{19}$$

Proof. The argument is analogous to that in the proof of Theorem 2.1, suitably adapted to account for jumps. Fix $p\in(0,1)$. Introducing the function $\psi(t) = \psi_{-\alpha^{-1}}(t) = \frac{1}{\varepsilon}\phi_\varepsilon(\varepsilon^{1/\alpha}t)$, by the definition in (15) and (17) we find that, similarly to (9),

$$\begin{aligned}
\varepsilon\psi(t) &= \frac{1}{\Gamma(\alpha)}\int_0^{\varepsilon^{1/\alpha}t}(\varepsilon^{1/\alpha}t - s)^{\alpha-1}\Big(\frac{1}{2}(p^2-p) + (\rho p\sigma-\lambda)\psi(\varepsilon^{-1/\alpha}s) + \frac{1}{2}\sigma^2\psi(\varepsilon^{-1/\alpha}s)^2 \\
&\qquad\qquad + V_\varepsilon(p,\varepsilon\psi(\varepsilon^{-1/\alpha}s))\Big)ds \\
&= \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\Big(\frac{1}{2}(p^2-p) + (\rho p\sigma-\lambda)\psi(u) + \frac{1}{2}\sigma^2\psi(u)^2 + V_\varepsilon(p,\varepsilon\psi(u))\Big)du\,,
\end{aligned}$$

where in the second equality we set $u = \varepsilon^{-1/\alpha}s$. But since $V_\varepsilon(p,\varepsilon\psi(\cdot)) = V_1(p,\psi(\cdot))$ (by (14)), the previous VIE is independent of ε and, by (15) with $\varepsilon=1$, we can write

$$\psi(t) \quad = \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}F_1(p,\psi(u))du, \quad t\ge 0.$$

In order to study the asymptotic behavior of ψ using Lemma 4.5 in [FGS21], we now verify that $F_1(p,\cdot)$ is (strictly) decreasing and analytic on $\mathbb{R}_-$. To this end, it is sufficient to show that $V_1(p,\cdot)$ is nonincreasing and analytic on the nonpositive half-line. By its definition in (13) with $\varepsilon = 1$, the map $V_1(p,\cdot)$ on $\mathbb{R}_-$ can be decomposed as follows:

$$\begin{aligned}
V_1(p,v) &= \int_{\mathbb{R}_+}\Big(e^{-\Lambda pz} - p(e^{-\Lambda z}-1) - 1\Big)\nu(dz) + \int_{\mathbb{R}_+} e^{-\Lambda pz}\Big(e^{vz} - 1 - vz\Big)\nu(dz) \\
&\quad + v\int_{\mathbb{R}_+} z(e^{-\Lambda pz}-1)\nu(dz),
\end{aligned} \tag{20}$$

for every $v\le 0$. This shows that $V_1(p,\cdot)$ is nonincreasing on $\mathbb{R}_-$, as it is given by the integral in $\nu(dz)$ of decreasing functions, namely $v\mapsto e^{vz}-1-vz$, for every $z\ge 0$, and a line with nonpositive slope. Moreover, recalling that $\Lambda>0$ and $p\in(0,1)$, differentiating $V_1(p,\cdot)$ in (20), we see that, for every $v\le 0$,

$$\begin{aligned}
V_1'(p,v) &= \int_{\mathbb{R}_+} z(e^{-\Lambda pz}-1)\nu(dz) + \int_{\mathbb{R}_+} ze^{-\Lambda pz}(e^{vz}-1)\nu(dz), \\
V_1^{(n)}(p,v) &= \int_{\mathbb{R}_+}|z|^n e^{vz-\Lambda pz}\nu(dz),\ n\ge 2.
\end{aligned}$$

Hence $V_1(p,\cdot)$ is of class C^∞ on $\mathbb{R}_-$. Notice that, for every $n\ge 2$, the function $z\mapsto |z|^{n-2}e^{-\Lambda pz}$ attains its maximum on $\mathbb{R}_+$ at $z = \frac{n-2}{\Lambda p}$, so

$$V_1^{(n)}(p,v) = \int_{\mathbb{R}_+}|z|^n e^{vz-\Lambda pz}\nu(dz) \le \Big(\int_{\mathbb{R}_+}|z|^2\nu(dz)\Big)n^n(\Lambda pe)^{2-n}, \quad v\le 0,\ \text{for all } n\ge 2.$$

Considering also that the first derivative of $V_1(p,\cdot)$ is locally bounded on $\mathbb{R}_-$, by Stirling's approximation of the factorial we deduce the analyticity of $V_1(p,\cdot)$ on $\mathbb{R}_-$.

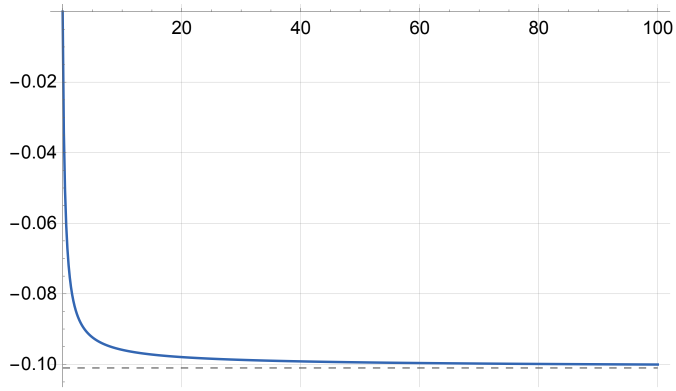

Figure 1: Here we see convergence of the VIE $\psi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} F_1(p, \psi(u))du$ to $U_J(p)$ defined in (18) as t increases. This convergence result is crucial for the proof of Proposition 3.1. Parameters: $\alpha = 0.7$, $\sigma = 1$, $\lambda = 1$, $\rho = -0.5$, $\Lambda = 1$, $p = \frac{1}{2}$ and $\nu(x) = \frac{Ce^{-Mx}}{x^{1+Y}} 1_{x>0}$, with $C = 0.1$, $M = 3$ and $Y = 1.5$. We use an Adams scheme with 500 time steps.

Therefore, the function $F_1(p, \cdot)$ is analytic and (strictly) decreasing on $\mathbb{R}_-$. Moreover, using the decomposition of $V_1(p, \cdot)$ in (20),

$$F_1(p, 0) = \frac{1}{2}(p^2 - p) + V_1(p, 0) = \frac{1}{2}(p^2 - p) + \int_{\mathbb{R}_+} \left(e^{-\Lambda pz} - p(e^{-\Lambda z} - 1) - 1\right)\nu(dz) < 0,$$

where the last inequality holds by (30) below and $p \in (0, 1)$. Furthermore, by straightforward estimates on $\mathbb{R}_-$ based on (15) and (20),

$$F_1(p, v) \geq F_1(p, 0) + (\rho p \sigma - \lambda)v \underset{v \to -\infty}{\longrightarrow} \infty.$$

These properties imply that $U_J(p)$ in (18) is well-defined. Hence another application of Lemma 4.5 in [FGS21] yields, for every fixed $t > 0$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \phi_\varepsilon(t) = \lim_{\varepsilon \to 0} \psi(\varepsilon^{-1/\alpha} t) = \lim_{s \to \infty} \psi(s) = U_J(p);$$

we refer to Figure 1 for a visualization of this convergence result. At this point, by the dominated convergence theorem (cf. (10)), we can take the limit as $\varepsilon \to 0$ in (16) to deduce (19), which completes the proof. ■

3.1 Characterizing the limiting law for X^ε in terms of a time-changed Lévy process with drift

In Theorem 3.2 below, we construct a time-changed Lévy process with drift $X = (X_t)_{t \geq 0}$ whose time-t marginal has log moment generating function $\lambda \theta U_J(p) t$, where $U_J(p)$ is defined in (18). By Proposition 3.1, this implies that the log price X_t^ε in a rescaled rough Hawkes–Heston model converges to X_t in distribution as $\varepsilon \to 0$. In Remark 3.1, we argue that Theorem 3.2 reduces to Theorem 2.1 in Section 2 – where the distributional limit is an NIG process – when there are no jumps, i.e., $\nu \equiv 0$. Finally, in Figure 2 we visualize the weak convergence result in Theorem 3.2 via IV smiles, and in Figure 3 we show the sensitivity of the IV smiles for the limiting model with log price process X with respect to the measure ν and the leverage parameter Λ.

Theorem 3.2 *Define the spectrally positive Lévy process $J = (J_t)_{t \geq 0}$ by*

$$J_t = \int_0^t \int_{\mathbb{R}_+} z\Big(N(dz, ds) - \frac{1}{\bar{\rho}^2}\nu(dz)ds\Big), \quad t \geq 0,$$

where $N(dz, dt)$ is a homogeneous Poisson random measure with intensity $\frac{1}{\bar{\rho}^2}\nu(dz) \otimes dt$ and $\bar{\rho} = \sqrt{1 - \rho^2} > 0$. Consider two standard Brownian motions $W = (W_t)_{t \geq 0}$ and $\tilde{W} = (\tilde{W}_t)_{t \geq 0}$, and suppose that J, W and $\tilde{W}$ are mutually independent. Set

$$\tilde{\beta} = \frac{1}{\bar{\rho}^2}\Big(\frac{\lambda\rho}{\sigma} - \frac{1}{2}\Big) - \frac{1}{\bar{\rho}^2}\int_{\mathbb{R}_+} (e^{-\Lambda z} - 1 + \Lambda z)\nu(dz), \qquad \gamma = \frac{\lambda}{\sigma\bar{\rho}}, \qquad \delta = \lambda\theta\frac{\bar{\rho}}{\sigma}, \qquad \mu = -\lambda\theta\frac{\rho}{\sigma}. \tag{21}$$

Define the time-changed Lévy process with drift $X = (X_t)_{t \geq 0}$ by

$$X_t = \mu t + \tilde{\beta}\tau_t + \tilde{W}_{\tau_t} - \Big(\Lambda + \frac{\rho}{\sigma}\Big)J_{\tau_t},$$
$$\text{where} \quad \tau_t := \inf\Big\{s \geq 0 : \gamma s + W_s - \frac{\bar{\rho}}{\sigma}J_s > \delta t\Big\}, \quad t \geq 0. \tag{22}$$

Then, for every $t > 0$, X_t has log moment generating function $t\lambda\theta U_J(\cdot)$ on $(0, 1)$, where $U_J(p)$ is defined in (18). Hence X_t^ε defined in (12) converges weakly to X_t as $\varepsilon \to 0$.

Remark 3.1 *Observe that the parameters γ, δ, and μ in (21) coincide with those in (7). In the no-jumps case, i.e., when $\nu \equiv 0$, the parameter $\tilde{\beta}$ in (21) also coincides with β in (7). In particular, by (4), the process X defined in (22) is an $NIG(\hat{\alpha}, \beta, \delta, \mu)$ Lévy process, where $\hat{\alpha} = \sqrt{\beta^2 + \gamma^2}$. Hence, in the continuous case $\nu \equiv 0$, Theorem 3.2 reduces to Theorem 2.1.*

Proof. By, for instance, Theorem 25.17 in [Sato99],

$$\mathbb{E}\big[e^{vJ_t}\big] = \exp\Big\{\frac{t}{\bar{\rho}^2}V_1(0, v)\Big\}, \quad v \le 0,\ t \ge 0, \tag{23}$$

where V_1 is defined in (13) with $\varepsilon = 1$. Define the Lévy process $\eta = (\eta_t)_{t\ge0}$ by

$$\eta_t = \tilde{\beta}t + \tilde{W}_t - \Big(\Lambda + \frac{\rho}{\sigma}\Big)J_t, \quad t \ge 0.$$

We also introduce, for every $p \in (0,1)$ and $u \le 0$, the process $(\xi_t(p,u))_{t\ge0}$ given by

$$\xi_t(p,u) = p\Big(\eta_t + \frac{\mu}{\delta}\Big(\gamma t + W_t - \frac{\bar{\rho}}{\sigma}J_t\Big)\Big) - \frac{u}{\delta}\Big(\gamma t + W_t - \frac{\bar{\rho}}{\sigma}J_t\Big), \quad t \ge 0. \tag{24}$$

Since $W, \tilde{W}$ and J are mutually independent, using the moment generating function of the normal distribution we compute

$$\begin{aligned}
&\log\mathbb{E}\big[\exp\{\xi_t(p,u)\}\big]\\
&\quad= \Big(p\tilde{\beta} + \frac{p\mu - u}{\delta}\gamma\Big)t + \log\mathbb{E}\Big[e^{p\tilde{W}_t}\Big] + \log\mathbb{E}\Big[\exp\Big\{\frac{p\mu - u}{\delta}W_t\Big\}\Big]\\
&\qquad+ \log\mathbb{E}\Big[\exp\Big\{\Big(\frac{\bar{\rho}}{\sigma\delta}u - \Lambda p\Big)J_t\Big\}\Big]\\
&\quad= \Big(p\tilde{\beta} + \frac{p\mu - u}{\delta}\gamma\Big)t + \frac{1}{2}\Big(p^2 + \frac{1}{\delta^2}(p^2\mu^2 + u^2 - 2\mu up)\Big)t + \frac{1}{\bar{\rho}^2}V_1\Big(0, \frac{\bar{\rho}}{\sigma\delta}u - \Lambda p\Big)t.
\end{aligned}$$

Here, in the first equality, to obtain the fourth term in J_t, we recall that η_t also depends on J_t and simplify two terms thanks to the relation $\mu/\delta = -\rho/\bar{\rho}$, which holds by (21). In the second equality, we apply (23). Then, considering that, by the definitions of $V_1(\cdot,\cdot)$ and $\tilde{\beta}$ in (13) and (21), respectively,

$$\begin{aligned}
&p\tilde{\beta} + \frac{1}{\bar{\rho}^2}V_1\Big(0, \frac{\bar{\rho}}{\sigma\delta}u - \Lambda p\Big)\\
&\quad= \frac{1}{\bar{\rho}^2}\Big[p\Big(\frac{\lambda\rho}{\sigma} - \frac{1}{2}\Big) + \int_{\mathbb{R}_+}\Big(e^{(\frac{\bar{\rho}}{\sigma\delta}u - \Lambda p)z} - p(e^{-\Lambda z} - 1 + \Lambda z) - 1 - \big(\frac{\bar{\rho}}{\sigma\delta}u - \Lambda p\big)z\Big)\nu(dz)\Big]\\
&\quad= \frac{1}{\bar{\rho}^2}\Big[p\Big(\frac{\lambda\rho}{\sigma} - \frac{1}{2}\Big) + V_1\Big(p, \frac{\bar{\rho}}{\sigma\delta}u\Big)\Big],
\end{aligned}$$

by algebraic manipulations based on (21), we complete the previous chain of equalities:

$$\log\mathbb{E}\big[\exp\{\xi_t(p,u)\}\big] = \frac{1}{\bar{\rho}^2}\Big[\frac{1}{2}(p^2 - p) + (\sigma\rho p - \lambda)\frac{u}{\lambda\theta} + \frac{1}{2}\sigma^2\Big(\frac{u}{\lambda\theta}\Big)^2 + V_1\Big(p, \frac{u}{\lambda\theta}\Big)\Big]t. \tag{25}$$

Recalling the map $F_1(p,\cdot)$ defined in (15) with $\varepsilon = 1$, we then have

$$\log\mathbb{E}\big[\exp\{\xi_t(p,u)\}\big] = \frac{t}{\bar{\rho}^2}F_1\Big(p, \frac{u}{\lambda\theta}\Big), \quad t \ge 0.$$

Choosing $u = \lambda\theta U_J(p)$ with $U_J(p)$ defined in (18), since $(\xi_t(p, \lambda\theta U_J(p)))_{t\ge0}$ has stationary and independent increments with respect to the natural filtration generated by the processes W, $\tilde{W}$ and J, the previous equation implies that $(\exp\{\xi_t(p, \lambda\theta U_J(p))\})_{t\ge0}$ is a martingale.

As in (22), for every $t \ge 0$, let τ_t denote the first hitting time to the upper barrier δt of the (spectrally negative) Lévy process $Y = (Y_s)_{s\ge0}$ defined by $Y_s = \gamma s + W_s - \frac{\bar{\rho}}{\sigma}J_s$, namely $\tau_t := \inf\big\{s \ge 0 : Y_s > \delta t\big\} = \inf\big\{s \ge 0 : \gamma s + W_s - \frac{\bar{\rho}}{\sigma}J_s > \delta t\big\}$. Note that $\mathbb{E}[Y_1] = \gamma = \frac{\lambda}{\sigma\bar{\rho}} > 0$, so $\tau_t < \infty$ $\mathbb{P}-$a.s. (see Theorem 36.5 in [Sato99], and also Proposition C.1 in [BF26]). Then, by the optional stopping theorem and the definition of $\xi_t(p, \lambda\theta U_J(p))$ in (24) we deduce that

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{\xi_{\tau_t\wedge n}(p, \lambda\theta U_J(p))\Big\}\Big]\\
&\quad= \mathbb{E}\Big[\exp\Big\{p\Big(\eta_{\tau_t} + \frac{\mu}{\delta}Y_{\tau_t}\Big) - \lambda\theta\frac{U_J(p)}{\delta}Y_{\tau_t}\Big\}1_{\{\tau_t\le n\}}\Big]\\
&\qquad+ \mathbb{E}\Big[\exp\Big\{p\Big(\eta_n + \frac{\mu}{\delta}Y_n\Big) - \lambda\theta\frac{U_J(p)}{\delta}Y_n\Big\}1_{\{\tau_t>n\}}\Big] = 1, \quad n \in \mathbb{N}.
\end{aligned} \tag{26}$$

Since $\lim_{n\to\infty}1_{\{\tau_t\le n\}} = 1_{\{\tau_t<\infty\}} = 1$, $\mathbb{P}-$a.s., and $Y_{\tau_t} = \delta t$ because Y is spectrally negative, from the monotone convergence theorem we see that the first term tends to

$$\mathbb{E}\Big[\exp\Big\{p(\mu t + \eta_{\tau_t}) - \lambda\theta U_J(p)t\Big\}\Big] = e^{-\lambda\theta U_J(p)t}\mathbb{E}\Big[e^{p(\mu t + \eta_{\tau_t})}\Big] \quad \text{as } n \to \infty. \tag{27}$$

As for the second term, our goal is to use Vitali's convergence theorem to prove that

$$\lim_{n\to\infty}\mathbb{E}\Big[\exp\Big\{p\Big(\eta_n+\frac{\mu}{\delta}Y_n\Big)-\lambda\theta\frac{U_J(p)}{\delta}Y_n\Big\}1_{\{\tau_t>n\}}\Big]=0. \tag{28}$$

To do this, we first observe that

$$\lim_{n\to\infty}\exp\Big\{p\Big(\eta_n+\frac{\mu}{\delta}Y_n\Big)-\lambda\theta\frac{U_J(p)}{\delta}Y_n\Big\}1_{\{\tau_t>n\}}=0,\quad \mathbb{P}-\text{a.s.}, \tag{29}$$

again because $\mathbb{P}(\tau_t=\infty)=0$. We then show that the sequence of random variables

$$\Big(\exp\Big\{p\Big(\eta_n+\frac{\mu}{\delta}Y_n\Big)-\lambda\theta\frac{U_J(p)}{\delta}Y_n\Big\}1_{\{\tau_t>n\}}\Big)_n \text{ is uniformly integrable.}$$

Since $p\in(0,1)$, we can fix a $q\in(1,\frac{1}{p})$. Observing that, by the definition of τ_t,

$$\exp\Big\{-q\lambda\theta\frac{U_J(p)}{\delta}Y_n\Big\}1_{\{\tau_t>n\}}\le e^{q\lambda\theta|U_J(p)|t},$$

we compute, for every $n\in\mathbb{N}$,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big\{q\Big(p\Big(\eta_n+\frac{\mu}{\delta}Y_n\Big)-\lambda\theta\frac{U_J(p)}{\delta}Y_n\Big)\Big\}1_{\{\tau_t>n\}}\Big]&\le e^{q\lambda\theta|U_J(p)|t}\mathbb{E}\Big[\exp\Big\{qp\Big(\eta_n+\frac{\mu}{\delta}Y_n\Big)\Big\}\Big]\\&=e^{q\lambda\theta|U_J(p)|t}\exp\Big\{\frac{1}{2\bar\rho^2}pq(pq-1)n+\frac{1}{\bar\rho^2}\Big(\int_{\mathbb{R}_+}\Big(e^{-\Lambda pqz}-pq(e^{-\Lambda z}-1)-1\Big)\nu(dz)\Big)n\Big\}\\&\le e^{q\lambda\theta|U_J(p)|t}\exp\Big\{\frac{1}{\bar\rho^2}\Big(\int_{\mathbb{R}_+}\Big(e^{-\Lambda pqz}-pq(e^{-\Lambda z}-1)-1\Big)\nu(dz)\Big)n\Big\}\le e^{q\lambda\theta|U_J(p)|t}.\end{aligned}$$

Here, for the equality on the second line we use the moment generating function at 1 of $qp(\eta_n+\frac{\mu}{\delta}Y_n)=\xi_n(pq,0)$ (see (25) with $u=0$ and pq instead of p), while for the inequalities on the third line we use, respectively, $pq<1$ and the elementary estimate $y^a\le(1-a)+ay$, $y\ge0$, $a\in(0,1)$, which, choosing $y=e^{-\Lambda z}$ and $a=pq$, ensures that

$$e^{-\Lambda pqz}-pq(e^{-\Lambda z}-1)-1\le0,\quad z\ge0. \tag{30}$$

The previous computations show that

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\Big|\exp\Big\{p\Big(\eta_n+\frac{\mu}{\delta}Y_n\Big)-\lambda\theta\frac{U_J(p)}{\delta}Y_n\Big\}\Big|^q1_{\{\tau_t>n\}}\Big]<\infty;$$

given that $q>1$, this is sufficient for uniform integrability. Since (29) implies convergence in probability, we can apply Vitali's convergence theorem to deduce (28).
Combining (26), (27) and (28), we then conclude that $\mathbb{E}\big[e^{p(\mu t+\eta_{\tau_t})}\big]=e^{\lambda\theta U_J(p)t}$. As $p\in(0,1)$, recalling the definition of the process $X=(X_t)_{t\ge0}$ in (22), the previous equation shows that $X_t=\mu t+\eta_{\tau_t}$ has log moment generating function $t\lambda\theta U_J(\cdot)$ on $(0,1)$. By Proposition 3.1, an application of Problem 30.4 on Page 573 in [Bill86] ensures that X^ε_t defined in (12) tends weakly to X_t, which completes the proof. ■

References

[AJ21] Abi Jaber, E., "Weak existence and uniqueness for affine stochastic Volterra equations with L^1-kernels", *Bernoulli, 27*(3), 1583-1615, 2021.

[AAR25] Abi Jaber, E., E. Attal and M. Rosenbaum, "From Hyper Roughness to Jumps as $H\to-\frac{1}{2}$", to appear in *The Annals of Applied Probability.*

[AAS26] Abi Jaber, E., E. Attal and A. Sojmark, "Volterra clocks and their pure-jump limits: hitting times of curved boundaries", *Preprint arXiv:2605.29073*, 2026.

[AC24] Abi Jaber, E. and N. De Carvalho, "Reconciling rough volatility with jumps", *SIAM Journal on Financial Mathematics, 15*(3), 785-823, 2024.

[ALP19] Abi Jaber, E., M. Larsson and S. Pulido, "Affine Volterra processes", *The Annals of Applied Probability, 29*(5), 3155-3200, 2019.

[Appl09] Applebaum, D., "*Lévy processes and stochastic calculus*", Cambridge University Press, 2009.

[Bill86] Billingsley, P., "*Probability and measure*", John Wiley&Sons, 2nd edition, 1986.

[BF26] Bondi, A. and M. Forde, "Lévy processes as weak limits of rough Heston models", to appear in *Bernoulli.*

[BG25] Bourgey, F. and J. Gatheral, "The SSR under Quadratic Rough Heston", *Available at SSRN 5239929*, 2025.

[BLP24] Bondi, A., G. Livieri and S. Pulido, "Affine Volterra processes with jumps", *Stochastic Processes and their Applications, 168*, 1042-64, 2024.

[BNPRS26] Bourgey, F., P.Noble, I.Petursson, M.Rosenbaum, and G.Szymanski, "The Quadratic Rough Heston+ Model for Short-Dated Options", preprint

[BP26] Bondi, A. and S. Pulido, "Feller's test for explosions of stochastic Volterra equations", *Electronic Journal of Probability, 31*(102), 1-41, 2026.

[BPS24] Bondi, A., S. Pulido and S. Scotti, "The rough Hawkes Heston stochastic volatility model", *Mathematical Finance, 34*(4), 1197-1241, 2024.

[BJ26] Bourdon, A. and T. Jeannin, "Moments in rough Bergomi and boundary attainment in rough Heston", *Preprint arXiv:2606.07482*, 2026.

[FGS21] Forde, M., S. Gerhold and B. Smith, "Small-time, large-time and $H \to 0$ asymptotics for the rough Heston model", *Mathematical Finance, 31*(1), 203-241, 2021.

[FGW26] Friesen, M., S. Gerhold and K. Wiedermann, "Boundary behaviour of the Volterra square-root process", *Preprint arXiv:2606.07290*, 2026.

[GJR20] Gatheral, J., P. Jusselin and M. Rosenbaum, "The quadratic rough Heston model and the joint S&P 500/VIX smile calibration problem", *Preprint arXiv:2001.01789*, 2020.

[JP22] Jacquier, A. and A. Pannier, "Large and moderate deviations for stochastic Volterra systems", *Stochastic Processes and their Applications, 149*, 142-187, 2022.

[JR16] Jaisson, T. and M. Rosenbaum, "Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes", *The Annals of Applied Probability, 26*(5), 2860-2882, 2016.

[JR20] Jusselin, P. and M. Rosenbaum, "No-arbitrage implies power-law market impact and rough volatility", *Mathematical Finance, 30*(4), 1309-1336, 2020.

[MORS26] Muhle-Karbe, J., Y. Chahdi, M. Rosenbaum and G. Szymanski, "A unified theory of order flow, market impact, and volatility", *Preprint arXiv:2601.23172*, 2026.

[Rom22] Rømer, S., "Hybrid multifactor scheme for stochastic Volterra equations with completely monotone kernels", *Available at SSRN 3706253*, 2022.

[Sato99] Sato, K., "*Lévy Processes and Infinitely Divisible Distributions*", Cambridge University Press, 1999.

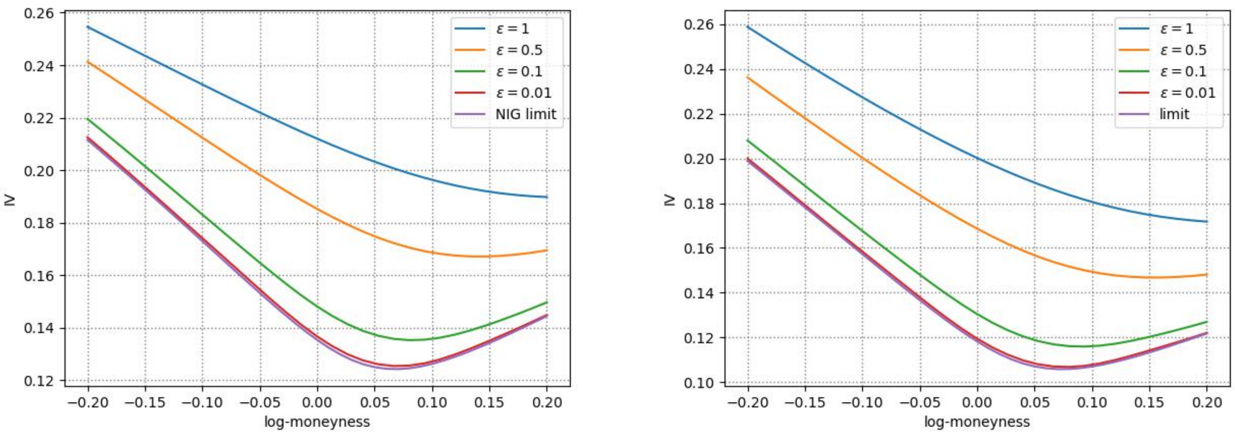

Figure 2: On the left, we have plotted IV smiles for European options with underlying price processes $e^{X_t^{\varepsilon}}$ and e^{X_t} defined in Theorem 2.1. Parameters: $S_0 = 1$, $\alpha = 0.7$, $\rho = -0.5$, $V_0 = 0.07$, $\sigma = \frac{1}{2}$, $\lambda = 1$, $\theta = 0.04$ and maturity $T = 1$. On the right, IV smiles for European options with underlying price processes $e^{X_t^{\varepsilon}}$ and e^{X_t} defined in Theorem 3.2, using the same parameters as before, with $\nu(dx) = e^{-x}dx$ and $\Lambda = 1$. Option prices are computed using the standard Lewis Fourier inversion formula. In both plots, as $\varepsilon \to 0$, the convergence of the IV smiles to the limiting theoretical smiles becomes apparent.

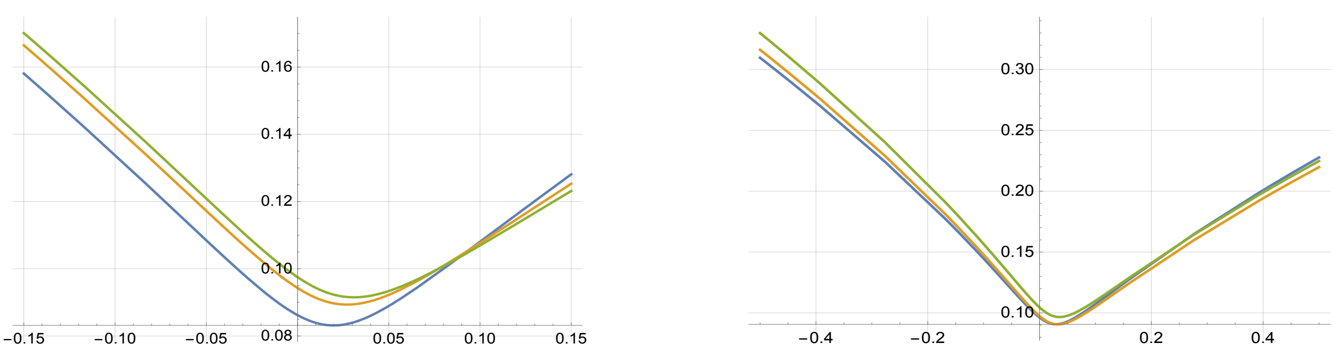

Figure 3: Here we have plotted IV smiles as functions of log-moneyness for the asymptotic model $S_t = e^{X_t}$ in Theorem 3.2, with $\rho = -0.5$, $\theta = 0.04$, $\lambda = 1$, $\sigma = 1$ and maturity $T = 1$. The measure $\nu(dx)$ is the same as in Figure 1, but with $M = 2$ and varying C. *Left:* $C = 1$ (blue), $C = 0.25$ (yellow) and $C = 0.01$ (green), with $\Lambda = 0$. *Right:* $C = 0.01$ fixed and $\Lambda = 0, 1, 2$ (blue, yellow and green, respectively).