The QGARCH(1,1) model

The QGARCH(1,1) model is a well known discrete-time model defined as

$$\begin{aligned} R_t &= \sqrt{V_t}\, Z_t \\ V_t &= \omega + \alpha R_{t-1}^2 + \beta V_{t-1} + \gamma R_{t-1} \end{aligned} \tag{1}$$

for $t = 1, 2...$ (e.g. days) where $R_t = (S_t - S_{t-1})/S_{t-1}$ is the t'th **stock price return** (note $R_t \geq -1$ since $S_t \geq 0$) and $\omega, \alpha, \beta > 0$, and Z_t is a sequence of i.i.d random variables with zero mean and variance σ^2, e.g. $N(0,1)$ or a student t-distribution with ν degrees of freedom if we want fatter tails for which $\sigma^2 = \frac{\nu}{\nu-2}$, so we need $\nu > 2$. Since we can re-write the model as

$$V_t = V_{t-1} + (1-\beta)(\bar{\omega} - V_{t-1}) + \alpha R_{t-1}^2 + \gamma R_{t-1} \tag{2}$$

where $\bar{\omega} = \frac{\omega}{1-\beta}$, we see that $1-\beta$ controls the **mean reversion** speed for V, and $\bar{\omega}$ is level around which V mean reverts. α controls the extent of **volatility clustering**, i.e. past large volatility giving rise to large future volatility and vice versa, and γ is a **skew term** which captures that squared volatility V_t tends to increase if $R_{t-1} < 0$ since usually $\gamma < 0$ as well so $\gamma R_{t-1} > 0$ (the so-called **leverage** effect). $\gamma < 0$ also allows the model to produce negatively skewed non-symmetric implied volatility smiles for European options which are seen in practice, particularly for Index and Equity options. The original Engle&Bollerslev **GARCH** model from 1986 has $\gamma = 0$, so the model above is sometimes known as the **asymmetric GARCH** model.

If we now instead say that V_{t+1} is V_t, then we can re-write the model in the **Euler-scheme** type form

$$\begin{aligned} S_t &= S_{t-1} + S_{t-1}\sqrt{V_{t-1}}\, Z_t \\ V_t &= V_{t-1} + (1-\beta)(\bar{\omega} - V_{t-1}) + \alpha R_t^2 + \gamma R_t \\ &= V_{t-1} + (1-\beta)(\bar{\omega} - V_{t-1}) + \alpha V_{t-1} Z_t^2 + \gamma \sqrt{V_{t-1}}\, Z_t \end{aligned} \tag{3}$$

then we see that (S_t, V_t) is **discrete-time Markov process**, since the distribution of S_t, V_t at time $t-1$ depends only on (S_{t-1}, V_{t-1}) and does not require any further history of these two processes (note our original V_t is now V_{t-1} here).

Taking expectations in (1), we see that

$$\mathbb{E}(V_t) = \omega + \alpha \mathbb{E}(R_{t-1}^2) + \beta \mathbb{E}(V_{t-1}) + \gamma \mathbb{E}(R_{t-1}) .$$

Using the **tower property** of conditional expectations, we can further re-write this as

$$\begin{aligned} \mathbb{E}(V_t) &= \omega + \alpha \mathbb{E}(\mathbb{E}(R_{t-1}^2)|V_{t-1}) + \beta \mathbb{E}(V_{t-1}) + \gamma \mathbb{E}(\mathbb{E}(R_{t-1}|V_{t-1})) \\ &= \omega + \alpha \mathbb{E}(\sigma^2 V_{t-1}) + \beta \mathbb{E}(V_{t-1}) + 0 \end{aligned} \tag{4}$$

where we have also used that $\mathbb{E}(R_{t-1}^2|V_{t-1}) = \mathbb{E}(V_{t-1}Z_{t-1}^2|V_{t-1}) = V_{t-1}\mathbb{E}(Z_{t-1}^2|V_{t-1}) = V_{t-1}\mathbb{E}(Z_{t-1}^2) = V_{t-1}\sigma^2$. For V_t to have a **stationary distribution**, i.e. for V_t to have the same distribution for all t, this clearly requires that $\mathbb{E}(V_t) = \mathbb{E}(V_{t-1})$, so we can further re-write (4) as

$$\mathbb{E}(V_t) = \omega + \alpha\sigma^2 \mathbb{E}(V_t) + \beta \mathbb{E}(V_t) .$$

and

$$\mathbb{E}(R_t^2) = \mathbb{E}(\mathbb{E}(R_t^2|V_t)) = \mathbb{E}(V_t) .$$

Re-arranging, we see that

$$\mathbb{E}(V_t) = \frac{\omega}{1 - \alpha\sigma^2 - \beta} .$$

Since V_t cannot be negative, we must have that $\alpha\sigma^2 + \beta < 1$, which we call the **stationarity condition**. If V starts at time zero, then

$$\begin{aligned} \mathbb{E}(V_t) &= \frac{1}{1-\alpha\sigma^2}(\omega + \beta \mathbb{E}(V_{t-1})) \\ \Rightarrow \quad \mathbb{E}(V_t) - \bar{V} &= \frac{1}{1-\alpha\sigma^2}(\omega + \beta \mathbb{E}(V_{t-1})) - \bar{V} = \frac{\beta}{1-\alpha\sigma^2}(\mathbb{E}(V_{t-1}) - \bar{V}) \end{aligned}$$

i.e. a **linear recurrence** relation of the form $r_t = a r_{t-1}$, with solution $r_t = \mathbb{E}(V_t) - \bar{V} = (\frac{\beta}{1-\alpha\sigma^2})^t (V_0 - \bar{V})$.

Moreover

$$V_t = \omega + \alpha R_{t-1}^2 + \beta V_{t-1} + \gamma R_{t-1} \geq \omega + \alpha R_{t-1}^2 + \gamma R_{t-1}$$

and (using basic calculus) the right-hand side is ≥ 0 for all R_{t-1} if $\omega \geq \frac{\gamma^2}{4\alpha}$. This is known as the **positivity condition**.

Let

$$\mathbb{E}(R_t^4) = \mathbb{E}(\mathbb{E}(R_t^4|\mathcal{F}_{t-1}) = \mathbb{E}(V_t^2\mathbb{E}(Z_t^4|\mathcal{F}_{t-1})) = \mathbb{E}(Z_t^4)\mathbb{E}(V_t^2)\,. \tag{5}$$

For $\gamma = 0$ and $\sigma = 1$, we have

$$\begin{aligned}\mathbb{E}(V_t^2) &= (3+K_\varepsilon)\mathbb{E}(V_t^2)\alpha^2 + 2\mathbb{E}(R_{t-1}^2 V_{t-1})\alpha\beta + \mathbb{E}(V_t^2)\beta^2 + 2\mathbb{E}(V_t)\alpha\omega + 2\mathbb{E}(V_t)\beta\omega + \omega^2 \\ &= (...) + 2\alpha\beta\mathbb{E}(V_{t-1}\mathbb{E}_{t-2}(R_{t-1}^2)) \\ &= (...) + 2\alpha\beta\mathbb{E}(V_t^2)\end{aligned}$$

Re-arranging the final expression, we see that

$$\mathbb{E}(V_t^2) = \frac{\omega(2\mathbb{E}(V_t)(\beta+\alpha)+\omega}{1-((3+K_\varepsilon)\alpha^2+\beta^2+2\alpha\beta)}\,.$$

if the denominator is positive.

Quasi Maximum likelihood estimates for the GARCH parameters and asymptotic normality

If V_1 is fixed and known and we start the model at time zero rather than $t = -\infty$, the joint density of $R_1, ..., R_n$ can be easily expressed as a product of conditional densities of the returns:

$$L = p(R_1)\,p(R_2|R_1)\,p(R_3|R_1,R_2)... = p(R_1)\,p(R_2|V_2)...\,p(R_n|V_n) = \prod_{j=1}^n f(\frac{R_j}{\sqrt{V_j}})\frac{1}{\sqrt{V_j}} = p(R_1)\,p(R_2|V_2)...\,p(R_n|V_n)$$

where f is the density of each Z_t in (1). This is true because

$$\mathbb{P}(R_j \leq x|V_j) = \mathbb{P}(Z_j \leq \frac{x}{\sqrt{V_j}}|V_j) = F(\frac{x}{\sqrt{V_j}})$$

where F is the distribution function of Z_t. Using *observed* values for $R_1, ..., R_n$, and given parameter values for the model, the values of $Z_j = \frac{R_j}{\sqrt{V_j}}$ are known as the **residuals** and L is the likelihood function of $R_1, ..., R_n$. We can then maximize L over all admissible parameter combinations to compute MLEs for the model parameters $\omega, \alpha, \beta, \gamma$, and the parameter(s) for the distribution of each Z_t (this is conceptually similar to Part 2).

Then the log likelihood is

$$\ell_n(\theta) = \sum_{t=1}^n \log f(\frac{R_t}{\sqrt{V_t}}) - \frac{1}{2}\log V_t$$

and recall that V_j actually depends on $R_1, ..., R_{j-1}$ and the model parameters which we collectively denote by θ. Then the Fisher information matrix when the residuals are i.i.d. $N(0,1)$ for the true stationary GARCH model is

$$\begin{aligned}I(\theta) = -\mathbb{E}(\frac{\partial^2}{\partial\theta^2}\ell_n(\theta))^2) &= \sum_{t=1}^n \mathbb{E}(-\frac{-2R_t^2+V_t(\theta)}{2V_t(\theta)^3}\frac{\partial V_t(\theta)}{\partial\theta_i}\frac{\partial V_t(\theta)}{\partial\theta_j} + (R_t^2 - V_t(\theta))V_t(\theta)\frac{\partial^2 V_t(\theta)}{\partial\theta_i\partial\theta_j}) \\ &= \sum_{t=1}^n \mathbb{E}(\frac{1}{2V_t(\theta)^2}\frac{\partial V_1(\theta)}{\partial\theta_i}\frac{\partial V_1(\theta)}{\partial\theta_j}) \\ &= n\mathbb{E}(\frac{1}{2V_1(\theta)^2}\frac{\partial V_1(\theta)}{\partial\theta_i}\frac{\partial V_1(\theta)}{\partial\theta_j})\end{aligned} \tag{6}$$

as $n \to \infty$, using the stationarity of V, where we have also used the tower property in the final line. For this to be useful we need to be able to sample from the stationary density for V_t, which we can approximate by considering n large for the model which starts at zero instead of $-\infty$. Then it can be shown that $\hat{\theta}_n$ is consistent and $\sqrt{n}(\hat{\theta}_n - \theta)$ tends to a multivariate $N(0, I(\theta)^{-1})$ random variable as $n \to \infty$, so intuitively we want parameters in the model such that $\frac{\partial V_1(\theta)}{\partial\theta_i}$ are larger.

Computing the stationary distribution for V

We note that

$$V_t \;\sim\; \omega + \alpha V Z + \beta V_t' \;=\; \omega + (\alpha Z + \beta)V' \;=\; \omega + A V_t'$$

where $A = \alpha Z^2 + \beta$, and $V_t' \sim V_t$. Conditioning on $A = a$, this implies that the stationary density $f_V(v)$ for V satisfies:

$$f_V(v) \;=\; \int_\beta^\infty f_V\Big(\frac{v-\omega}{a}\Big)\frac{1}{a}p_A(a)da \tag{7}$$

where $p_A(a)$ is density of A, and note A is just a linear transformation of a χ^2 random variable when $Z \sim N(0,1)$. (7) is a **linear Fredholm integral equation** for $f_V(v)$, which in principle can be solved by discretizing it and solving a linear system of equations.

Goodness-of-fit tests for the residuals

If e.g. we assume $Z_t \sim N(0,1)$, we can then perform standard normality tests like **Kolmogorov Smirnov**, **Shapiro-Wilk**, **Jarque-Bera** or **Andersen-Darling** to test whether the Z_t values are indeed i.i.d. Normals. Otherwise, if we use a different distribution for Z_t (e.g. a ***t*-distribution** with ν degrees of freedom which will give the returns fatter tails), we have to transform these back Z values to Normal RVs before applying these normality tests, using inverse cdfs.

Estimating V_0 from the stock price history

If we assume $\gamma = 0$ for simplicity, then iterating the definition of V_t we see that

$$\begin{aligned}
V_t &= \omega + \beta V_{t-1} + \alpha R_{t-1}^2 \\
&= \omega + \beta(\omega + \beta V_{t-2} + \alpha R_{t-2}^2) + \alpha R_{t-1}^2 \\
&= \omega + \beta(\omega + \beta(\omega + \beta V_{t-3} + \alpha R_{t-3}^2) + \alpha R_{t-2}^2) + \alpha R_{t-1}^2 \\
&= \omega(1+\beta+\beta^2+...) + \frac{\alpha}{\beta}\sum_{\tau=1} \beta^\tau R_{t-\tau}^2 \;=\; \bar{\omega} + \frac{\alpha}{\beta}\sum_{\tau=1}^\infty e^{-b\tau}R_{t-\tau}^2
\end{aligned} \tag{8}$$

where b is defined by $\beta = e^{-b}$ and $\bar{\omega}$ is defined above, and note the first term on the right-hand side is the mean reversion level from above. So we see that the effect of past returns on volatility decays exponentially, and re-doing this computation with $\gamma \neq 0$, we find that the last line just changes to

$$V_t \;=\; \frac{\omega}{1-\beta} + \frac{\alpha}{\beta}\sum_{\tau=1}^\infty e^{-b\tau}R_{t-\tau}^2 + \frac{\gamma}{\beta}\sum_{\tau=1}^\infty e^{-b\tau}R_{t-\tau}\,.$$

In particular, we also see that

$$V_0 \;=\; \bar{\omega} + \frac{\alpha}{\beta}\sum_{\tau=1}^\infty e^{-b\tau}R_{-\tau}^2 + \frac{\gamma}{\beta}\sum_{\tau=1}^\infty e^{-b\tau}R_{-\tau}$$

so we can estimate V_0 by truncating this sum in practice rather than fitting V_0 as an additional free parameter for the MLE maximization computation described above, since V_0 is already fixed by the history of the returns.

Stochastic volatility as the diffusive limit of QGARCH

Consider the following variant of the model above:

$$\begin{aligned}
S_t &= S_{t-\Delta t} + S_{t-\Delta t}\sqrt{V_{t-\Delta t}}\,Z_t \\
V_t &= V_{t-\Delta t} + \kappa\theta\Delta t + \frac{\eta}{\sqrt{\Delta t}}(R_t^2 - V_{t-\Delta t}\Delta t) - \kappa V_{t-\Delta t}\Delta t + \gamma R_t \\
&= V_{t-\Delta t} + \kappa(\theta - V_{t-\Delta t})\Delta t + \frac{\eta}{\sqrt{\Delta t}}V_{t-\Delta t}(Z_t^2 - \Delta t) + \gamma\sqrt{V_{t-\Delta t}}Z_t \\
&= V_{t-\Delta t} + \bar{\kappa}(\bar{\theta} - V_{t-\Delta t})\Delta t + \frac{\eta}{\sqrt{\Delta t}}R_t^2 + \gamma R_t
\end{aligned}$$

for some $\bar{\kappa}$, $\bar{\theta}$, with $Z_1, Z_2, ...$ i.i.d. as above and V_{t-1} here is our old V_t, and now assume $\text{Var}(Z_t) = \Delta t$ and $\eta = O(1)$, and impose that $\nu > 4$ so $\mathbb{E}(Z_i^4) < \infty$, and from the final line we see that V_t is still of the QGARCH(1,1) form in (3). Then as $\Delta t \to 0$, the model tends to the mean-reverting **Markov stochastic volatility** model:

$$\begin{aligned} dS_t &= S_t\sqrt{V_t}dW_t \\ dV_t &= \kappa(\theta - V_t)dt + 2\eta V_t dB_t + \gamma\sqrt{V_t}dW_t \end{aligned} \tag{9}$$

where W and B are standard independent Brownian motions, so we see that the specific form of the distribution of the Z_t's does not show up in the $\Delta t \to 0$ limit and the independent Brownian motion B appears almost by magic. When η is larger, the implied volatility smile will be more U-shaped as a function of strike K, and will be symmetric as a function of $x = \log\frac{K}{S_0}$ if $\gamma = 0$. If ν is smaller, the smile may just be monotonically decreasing as a function of K over relevant strike ranges.

The limiting model in (9) is hybrid of the well known **Hull-White** and **Heston** models (the well known Heston model has a $\sqrt{V_t}$ term in it). To see why this is true, we first note that

$$\frac{1}{\sqrt{\Delta t}}\sum_{i=1}^{[nt]}(Z_i^2 - \Delta t) = \sqrt{n}\sum_{i=1}^{[nt]}(\Delta t\tilde{Z}_i^2 - \Delta t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}(\tilde{Z}_i^2 - 1) \tag{10}$$

where $\tilde{Z}_i = Z_t/\sqrt{\Delta t} \sim N(0,1)$, and that $\text{Var}(\tilde{Z}_i^2 - 1) = \mathbb{E}((\tilde{Z}_i^2 - 1)^2) = 3 - 2 + 1 = 2$.

We now recall **Donsker's theorem**. Let X_i be a sequence of i.i.d. random variables with $\mathbb{E}(X_i) = 0$ and $\text{Var}(X_i) = 1$, and let $S_n = \sum_{i=1}^n X_i$. Now consider the **random function**:

$$W_t^n = \frac{S_{[nt]}}{\sqrt{n}} \qquad (t \in [0,1])$$

where $[nt]$ denotes the largest integer less than or equal to nt. Then by the **Central Limit Theorem**, $W_1^n = \frac{S_n}{\sqrt{n}}$ tends to an $N(0,1)$ random variable as $n \to \infty$. More precisely, $\lim_{n\to\infty}\mathbb{E}(F(W_1^n)) = \mathbb{E}(F(Z))$ for any bounded continuous function F (this is known as **weak convergence**). Donsker's theorem, states that the random function W_t^n tends weakly to a random function which is a Brownian motion as $n \to \infty$. This shows that we can numerically approximate Brownian motion using X_i's with any distribution with finite variance. Thus (10) falls exactly under the framework of Donsker's theorem, aside from $\tilde{Z}_i^2 - 1$ having a variance of 2 not 1, which is why there is a **factor of** 2 in (9).

Changing from $\mathbb{P}$ to $\mathbb{Q}$ measure

If the Z_t's have a non-zero density under $\mathbb{P}$, then the Z_t's can have any non-zero density under $\mathbb{Q}$ (does not have to be equal to the original density), so long as $\mathbb{E}^{\mathbb{Q}}(Z_t) = 0$, then S will still be a martingale under $\mathbb{Q}$, which is equivalent to $\mathbb{P}$ since both densities are non-zero by assumption.

Intraday dynamics consistent with the QGARCH model

The t-distribution is infinitely divisible which means a random variable Z with this distribution can be written as a sum of n i.i.d random variables Z_i^n, for any n. The characteristic function $\mathbb{E}(e^{iuZ_i^n})$ of Z_i^n is then $\phi(u)^{1/n}$ where $\phi(u) = \mathbb{E}(e^{iuZ})$. This gives us a way to extend the model from modelling daily returns to intraday returns with n i.i.d residuals per day, keeping V constant within any given day.

Bayesian analysis

If we set $X = (R_1, ..., R_n)$ and $\theta = (\alpha, \beta, \gamma, \nu)$, then from Bayes formula, we know that

$$p(\theta|X) = \frac{p(X|\theta)\,p(\theta)}{p(X)}$$

where the p's refer to densities or conditional densities here. $p(X)$ does not depend on θ, and if assume a uniform prior $p(\theta) = const.$ for θ on some finite hypercube in $\mathbb{R}^4$ (and zero elsewhere), then

$$p(\theta|X) = const. \times p(X|\theta)$$

so the conditional density of θ given X is proportional to the likelihood function $p(X|\theta)$, and by integrating in the other 3 parameters we can compute e.g. the marginal density of α, β, γ or ν given X. This is easier if e.g. we fix $\gamma = 0$ and fix $1 - \beta$ to its lower bound, so we only have two free parameters.

Power kernel model

We can modify the model as follows:

$$
\begin{aligned}
R_t &= \sqrt{V_t}\, Z_t \\
V_t &= \omega + c\sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau}^2 + \gamma \sum_{\tau=1}^{\infty} \tau^{-\alpha_2} R_{t-\tau}
\end{aligned}
$$

for $\alpha, \alpha_2 > 2$ (add mean reversion?) which corresponds to **power decay**, and again we have to take care to ensure positivity and stationarity. In this case, using the same tower law argument as above

$$
\mathbb{E}(V_t) = \omega + c\sum_{\tau=1}^{\infty} \tau^{-\alpha}\mathbb{E}(R_{t-\tau}^2) = \omega + c\sum_{\tau=1}^{\infty} \tau^{-\alpha}\mathbb{E}(\mathbb{E}(R_{t-\tau}^2|V_{t-\tau}) = \omega + c\sigma^2 \sum_{\tau=1}^{\infty} \tau^{-\alpha}\mathbb{E}(V_{t-\tau})\,.
$$

If V is stationary, then

$$
\mathbb{E}(V_t) = \omega + c\sigma^2 \sum_{\tau=1}^{\infty} \tau^{-\alpha}\mathbb{E}(V_t) = \omega + c\sigma^2\mathbb{E}(V_t)\zeta(\alpha)
$$

which we can re-arrange as $\mathbb{E}(V_t) = \frac{\omega}{1-c\sigma^2\zeta(\alpha)}$, where $\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha}$ denotes the **zeta function**, so clearly a necessary condition for stationarity is that $c\sigma^2\zeta(\alpha) < 1$.

If $\alpha = \alpha_2$, then can re-write as

$$
V_t = \sum_{\tau=1}^{\infty} \tau^{-\alpha}(\bar{\omega} + cR_{t-\tau}^2 + \gamma R_{t-\tau})
$$

where $\bar{\omega} = \frac{\omega}{\zeta(a)}$, so we have essentially the same **positivity condition** as before $\bar{\omega} \geq \frac{\gamma^2}{4c}$. This is a discrete-time version of the **rough Heston model**.

Quadratic Rough Heston-type model

We can also generalize to a quadratic rough Heston-type model:

$$
V_t = \omega + c\sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau}^2 + b(\sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau} - a)^2 + \gamma \sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau}\,.
$$

Then again assuming stationarity, we now see that

$$
\begin{aligned}
\mathbb{E}(V_t) &= \omega + c\sigma^2\zeta(\alpha)\mathbb{E}(V_t) + b\,\mathbb{E}(\sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau} - a)^2 \\
&= \omega + c\sigma^2\zeta(\alpha)\mathbb{E}(V_t) + b\mathbb{E}(\sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau})^2 + a^2) \\
&= \omega + c\sigma^2\zeta(\alpha)\mathbb{E}(V_t) + b(\zeta(2\alpha)\mathbb{E}(V_t) + a^2)
\end{aligned}
$$

using that $\mathbb{E}(R_i R_j) = \mathbb{E}(R_i\mathbb{E}(R_j|R_i, V_j)) = 0$ for $i < j$, so the stationarity condition now reads as $c\sigma^2\zeta(\alpha) + b(\zeta(2\alpha) < 1$.

Numerical results

Below we compute MLEs and apply the Kolmogorov Smirnov, Shapiro-Wilk and Jarque-Bera normality tests on the (transformed) residuals implied by the MLEs for the model in (1) using daily prices, with a 1yr/3yr/1yr test window (the initial 1yr window is used to compute the V_0 for the middle window from the initial 1yr history of returns; the middle 3yr period is used for in-sample (i/s) testing, and final year used for out-of-sample testing, all three periods are consecutive with no gaps/overlap), ending 11/08/2023. Although the fits are very good, the sample variance of the MLEs using synthetic paths with the fitted parameters are much higher than we would ideally like.

MLEs/p-vals	α	β	γ	ν	KS i/s	SW i/s	JB i/s	KS o/s	SW o/s	JB o/s
EUR/USD	0.0293	0.962	-5.405e-05	8.684	0.835	0.870	0.706	0.912	0.714	0.643
GBP/USD	0.0303	0.932	-0.000252	6.192	0.966	0.836	0.712	0.119	0.224	0.279
USD/JPY	0.0830	0.875	-0.000299	5.9611	0.292	0.476	0.352	0.0603	0.0907	0.229
AMZN	0.03482	0.9420	-0.000505	5.008	0.401	0.811	0.951	0.560	0.607	0.570
BRK-B	0.103	0.868	-0.00103	8.929	0.168	0.921	0.950	0.611	0.676	0.984
INTC	0.0280	0.943	-5.940e-05	3.914	0.375	0.0634	0.0404	0.229	0.262	0.236
AZN	0.0496	0.904	-0.000897	4.153	0.247	0.587	0.428	0.103	0.206	0.195
N225	0.0982	0.856	-0.00129	6.271	0.281	0.443	0.349	0.0713	0.236	0.354
HSI	0.06222	0.898	-0.000834	5.108	0.491	0.226	0.358	0.530	0.121	0.161

To fix SPX historical prices well, we need a skewed t-distribution for the residuals